1 First-Order Differential Equations

1.1 The Orthogonal Trajectory Problem

Let $F(x, y, c) = 0$ be the family of functions $y - \frac{c}{x} = 0$.

(a) Sketch the family functions:

(b) Find an associated family of curves, denoted $G(x, y, k) = 0$, which have the property that whenever a curve from the first family intersects a curve from this family, it does so at a right angle. These curves are called the orthogonal trajectories of the curves in (a) and vice versa.

(c) Sketch the orthogonal trajectories on the same graph as you did the original family of graphs.
1.2 Basic Ideas and Terminology

**Definition.** A differential equation (D.E.) is an equation involving one or more derivatives of an unknown function.

**Definition.** The order of the highest derivative occurring in a differential equation is called the **order** of the differential equation.

**Definition.** A differential equation that can be written in the form

\[ a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = F(x), \]

where \( a_0, a_1, \ldots, a_n \) and \( F \) are functions of \( x \) only, is called a **linear differential equation** of order \( n \). Such a differential equation is linear in \( y, y', y'', \ldots, y^{(n)} \).
**Definition.** A function \( y = f(x) \) that is (at least) \( n \) times differentiable on an interval \( I \) is called a **solution** to the differential equation on \( I \) if the substitution \( y = f(x), \ y' = f'(x), \ldots, y^{(n)} = f^{(n)}(x) \) reduces the differential equation to an identity valid \( \forall x \in I \). In this case, we say that \( y = f(x) \) satisfies the differential equation.

**Example 1.** Verify that \( y(x) = c_1 x^{1/2} + 3x^2 \) is a solution to \( 2x^2 y'' - xy' + y = 9x^2 \), where \( c_1 \) and \( c_2 \) are arbitrary constants, and state the maximum interval over which the solution is valid.

**Definition.** A solution to an \( n \)th-order differential equation on an interval \( I \) is called the **general solution on \( I \)** if it satisfies the following conditions:

1. The solution contains \( n \) constants \( c_1, c_2, \ldots, c_n \).
2. All solutions to the differential equation can be obtained by assigning appropriate values to the constants.

**Definition.** A solution to a differential equation which does not contain any arbitrary constants not present in the differential equation itself is called a **particular solution**.

**Definition.** An **initial value-problem** is an \( n \)th-order differential equation together with \( n \) auxiliary conditions of the form

\[
y(x_0) = y_0, \ y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1},
\]

where \( y_0, y_1, \ldots, y_{n-1} \) are constants.
Theorem 1.2.1. Let $a_1, a_2, \ldots, a_n, F$ be functions that are continuous on an interval $I$. Then, for any $x_0 \in I$, the initial value problem

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x),$$

$$y(x_0) = y_0, \ y'(x_0) = y_1, \ \ldots, \ y^{(n-1)}(x_0) = y_{n-1},$$

where $y_0, y_1, \ldots, y_{n-1}$ are constants, has unique solutions on $I$.

Note. The initial value-problem can be written in its normal form, namely

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$$

subject to $y(x_0) = y_0, \ y'(x_0) = y_1, \ \ldots, y^{(n-1)}(x_0) = y_{n-1}$ where $y_0, y_1, \ldots, y_{n-1}$ are constants. The theorem above says that this problem always has a unique solution provided that $f$ and its partial derivatives with respect to $y, y', \ldots, y^{(n-1)}$ are continuous in an appropriate region.

Example 2. Determine all values of the constant $r$ such that $y(x) = x^r$ solves $x^2y'' + 5xy' + 4y = 0$.

Example 3. Find the general solution to $y'' = \cos x$ and find the maximum interval on which the solution is valid.
Example 4. Find the particular solution to $y'' = -2(3 + 2 \ln x)$, given $y(1) = y(e) = 0$. 
1.4 Separable D.E.’s

**Definition.** A first-order differential equation that can be written in the form

\[
p(y) \frac{dy}{dx} = q(x)
\]

is called separable.

**Theorem 1.4.1.** If \( p(y) \) and \( q(x) \) are continuous, then \( p(y) \frac{dy}{dx} = q(x) \) has the general solution

\[
\int p(y)dy = \int q(x)dx + c
\]

where \( c \) is an arbitrary constant.

**Example 5.** Solve each differential equation:

(a) \( \frac{dy}{dx} = x e^{x^2 - \ln y^2} \)

(b) \( \frac{dy}{dx} = \frac{x^2 y - 4y}{x + 2} \)
(c) $x \frac{dy}{dx} = 2(y - 4)$

**Definition.** A slope field is a graphical representation of the solutions of a first order differential equation.

The lines represent the slopes of the function at $(x, y)$ - values and the curves are the actual functions whose slopes are those represented on the slope field.

**Note.** There are plenty websites that can help you graph the slope field. Here are some helpful ones:

- [https://www.desmos.com/calculator/p7vd3cdmei](https://www.desmos.com/calculator/p7vd3cdmei)
- [https://www.geogebra.org/m/QXhfbs79](https://www.geogebra.org/m/QXhfbs79)
- [http://bluffton.edu/homepages/facstaff/nesterd/java/slopefields.html](http://bluffton.edu/homepages/facstaff/nesterd/java/slopefields.html)

*If you have a Mac, you can use the “Grapher” embedded in every Mac. If you need help with it, let me know.*
Example 6. Find the equation of the curve that passes through the point \( (0, \frac{1}{2}) \) and whose slope at each point \( (x, y) \) is \( -\frac{x}{4y} \).
1.6 First-Order Linear Differential Equations

**Definition.** A differential equation that can be written in the form

\[ a(x) \frac{dy}{dx} + b(x)y = r(x) \]

where \( a(x), b(x), \) and \( r(x) \) are functions defined on an interval \((a, b)\), is called a **first-order linear differential equation**.

The **standard form** is

\[ \frac{dy}{dx} + p(x)y = q(x) \]

where \( p(x) = \frac{b(x)}{a(x)} \) and \( q(x) = \frac{r(x)}{a(x)} \), where \( a(x) \neq 0 \) on \((a, b)\).
Steps for Solving First-ODE’s

1. Put the first-order into standard form, \( \frac{dy}{dx} + p(x)y = q(x). \)

2. Find the (constant-less) integrating factor, \( I(x) = e^{\int p(x)dx}. \)

3. Multiply the entire equation by the integrating factor, \( I(x) \), to get the LHS to be \( \frac{d}{dx}[I(x)y]. \)

4. Integrate both sides.

5. Solve for \( y \) (or \( y(x) \)).

Example 7. Solve for the differential equation.

(a) \( y' = \sin x(y \sec x - 2) \)
(b) $y' + mx^{-1}y = \ln x$, where $m$ is a constant.
Example 8. Suppose that an object is placed in a medium whose temperature is increasing at a constant rate of $\alpha^\circ F$ per minute. Show that, according to Newton’s Law of Cooling, the temperature of the object at time $t$ is given by $T(t) = \alpha(t - k^{-1}) + c_1 + c_2e^{-kt}$ where $c_1$ and $c_2$ are constants.
1.7 Modeling Problems Using First-Order-Linear Differential Equations - Electric Circuits

**Note.** We are simplifying everything in this course. Thus, we are assuming “ideal” conditions.

**Definition.** An electric circuit is a network of a closed loop, giving a return path for the current.

**Definition.** An electric charge is the physical property of matter that causes it to experience a force when placed in an electromagnetic field. There are two types of electric charges: positive and negative. Like charges repel and unlike attract.

**Definition.** An electric current, denoted $i(t)$, is the flow rate in an electrical circuit and has unit measured in amperes, or amps ($A$). It is measured by the rate of electric charge flow in an electrical circuit.

*i.e.* Think of the electrical current as water current that flows in a pipe.

### Formula for Electrical Current

$$i(t) = \frac{dq}{dt},$$

where $i(t)$ is the momentary electrical circuit $i$ at time $t$, $q(t)$ is the momentary electric charge in coulombs ($C$) at time $t$, and $t$ is the time in seconds.

### Components and their Formulas

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<th>Definition</th>
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<th>Formula</th>
<th>Formula Meaning</th>
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<tr>
<td>resistor</td>
<td>an electrical component that resist, or reduces current flow</td>
<td>ohms ($\Omega$)</td>
<td>$\Delta V_R = iR$</td>
<td>The voltage drop, $\Delta V_R$, between the ends of a resistor is directly proportional to the current that is passing through it where $i$ is the current and $R$ is the <strong>resistance</strong> of the resistor.</td>
</tr>
<tr>
<td>capacitor</td>
<td>an electrical component used to store charge temporarily</td>
<td>farads ($F$)</td>
<td>$\Delta V_C = \frac{1}{C}q$</td>
<td>The voltage drop, $\Delta V_C$, as current passes through a capacitor is directly proportional to $q(t)$ which is the charge on the capacitor at time $t$ where $C$ is the <strong>capacitance</strong> of the capacitor.</td>
</tr>
<tr>
<td>inductor</td>
<td>an electrical component which resists changes in electric current passing through it</td>
<td>henrys ($H$)</td>
<td>$\Delta V_L = L\frac{di}{dt}$</td>
<td>The voltage drop, $\Delta V_L$, as current passes through an inductor is directly proportional to the rate at which the current is changing where $i$ is the current and $L$ is the <strong>inductance</strong> of the inductor.</td>
</tr>
</tbody>
</table>
**Definition.** Electromotive Force (EMF), denoted $E(t)$, the voltage developed by any source of electrical energy (like a battery). It is also called the **electrical potential** in a circuit. The **EMF** is measured in volts ($V$).

**Note.** There are two types of current:

1. Direct (*DC*): $E(t) = E_0$, with $E_0 \in \mathbb{R}$.
2. Alternating (*AC*): $E(t) = E_0 \cos(\omega t)$ with $\omega, E_0 \in \mathbb{R}$.

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**Kirchhoff’s Second Law**

The sum of the voltage drops around a closed circuit is zero. In other words, around any loop in a circuit, whatever energy a charge starts with, it must end up losing all of that energy by the time it gets to the end. Mathematically, this is equivalent to

$$\Delta V_R + \Delta V_C + \Delta V_L - E(t) = 0.$$
Circuit Cases:

1. RL Circuit: This is the case when there is no capacitor.

Example 9. What is the resulting differential equation for the RL circuit case?

Example 10. An RL circuit has an EMF = 5V, a resistance of 50Ω, an inductance of 1H, and no initial current. Find the current in the circuit at any time $t \geq 0$.

Note. $i(t) = i_S(t) + i_T(t)$ where $i_S(t)$ is the steady-state solution and $i_T(t)$ is the transient part of the solution. $i(t)$ approaches $i_S(t)$ as $t \to \infty$ since $\lim_{t \to \infty} i_T(t) = 0$. 
2. *RC* Circuit: This is the case when there is no inductor.

**Example 11.** What is the resulting equation for the *RC* circuit case?

**Example 12.** Find the charge and the current for \( t > 0 \) in an *RC* circuit where \( R = 10\Omega \), \( C = 4 \times 10^{-3} F \), and \( E(t) = 85 \cos(150t) V \). Assume that \( q(0) = -0.05C \). For the current, find the steady-state solution and the transient part of the solution as well.
1.8 Change of Variables

**Definition.** A function $f(x, y)$ is said to be homogeneous of degree zero if

$$f(tx, ty) = f(x, y)$$

for all positive values of $t$ for which $(tx, ty)$ is in the domain of $f$.

$$f(x, y) = \frac{y}{x}$$

is homogeneous of degree zero since

$$f(tx, ty) = \frac{ty}{tx} = \frac{y}{x} = f(x, y)$$

**Theorem 1.8.1.** A function $f(x, y)$ is homogeneous of degree zero if and only if it depends on $\frac{y}{x}$ only.

**Proof:**

**Definition.** If $f(x, y)$ is homogeneous of degree zero, then the differential equation

$$\frac{dy}{dx} = f(x, y)$$

is a homogeneous first-order differential equation.
Theorem 1.8.2. The change of variables \( y = xV(x) \) reduces a homogeneous first-order differential equation \( \frac{dy}{dx} = f(x, y) \) to the separable equation

\[
\frac{1}{F(V) - V} dV = \frac{1}{x} dx.
\]

Example 13. Solve the differential equation.

(a) \( x^2 \frac{dy}{dx} = y^2 + 3xy + x^2 \)
(b) \( \frac{dy}{dx} = \frac{x\sqrt{x^2 + y^2} + y^2}{xy} \) where \( x > 0. \)
**Definition.** A Bernoulli Equation is a differential equation that can be written in the form

\[
\frac{dy}{dx} + p(x)y = q(x)y^n,
\]

where \( n \in \mathbb{R} \). We use the change of variables \( u = y^{1-n} \) for this special type of equation.

**Example 14.** Solve \( \frac{dy}{dx} + \frac{2x}{1 + x^2} y = xy^2 \) where \( y(0) = 1 \).
1.9 Exact Differential Equations

**Definition.** \( M(x, y) \, dx + N(x, y) \, dy = 0 \) is exact (differential equation) in a region \( R \) of the \( xy \)-plane provided there exists a function \( \phi(x, y) \) such that

\[
\frac{\partial \phi}{\partial x} = M \quad \text{and} \quad \frac{\partial \phi}{\partial y} = N \quad \forall \, (x, y) \in R.
\]

In this case, \( \phi \) is called a **potential function** for the differential equation.

**Example 15.** \((2xy - 9x^2) \, dx + (2y + x^2 + 1) \, dy = 0\) is exact since for \( \phi(x, y) = y^2 + (x^2 + 1)y - 3x^3 \),

\[
\frac{\partial \phi}{\partial x} = 2xy - 9x^2 \quad \text{and} \quad \frac{\partial \phi}{\partial y} = 2y + x^2 + 1.
\]

**Theorem 1.9.1.** The general solution to an exact equation

\[
M(x, y) \, dx + N(x, y) \, dy = 0
\]

is defined implicitly by

\[
\phi(x, y) = c,
\]

where \( \phi \) is a potential function for the differential equation and \( c \) is an arbitrary constant.

**Proof:**
**Theorem 1.9.2. Test for Exactness** Let \( M, N \) and their first partial derivatives, \( M_y \) and \( N_x \), be continuous in a simply connected region \( R \) of the \( xy \)-plane. Then the differential equation

\[
M(x, y) \, dx + N(x, y) \, dy = 0
\]

is exact \( \forall x, y \in R \iff \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \)

*simply connected* is a path-connected domain where you can continuously shrink any simple closed curve into a point. For two-dimensional regions, it is a domain without holes in it.

**Example 16.** Solve \((2xy + \cos y) \, dx + (x^2 - x \sin y - 2y) \, dy = 0\) by first showing that it is exact.
**Definition.** A nonzero function $I(x, y)$ is called an **integrating factor** for the differential equation $M(x, y) \, dx + N(x, y) \, dy = 0$ provided the differential equation

$$I(x, y) \, M(x, y) \, dx + I(x, y) \, N(x, y) \, dy = 0$$

is exact.

**Example 17.** Show that $I(x, y) = y^{-2}e^{-x/y}$ is an integrating factor for $y(x^2 - 2xy) \, dx - x^3 \, dy = 0$. 
Example 18. (a) Determine the values of the constants \( r \) and \( s \) such that \( I(x, y) = x^r y^s \) is an integrating factor for \( y(5xy^2 + 4) \, dx + x(xy^2 - 1) \, dy = 0. \)
(b) Solve $y(5xy^2 + 4) \, dx + x(xy^2 - 1) \, dy = 0$. 
Theorem 1.9.3. The function $I(x, y)$ is an integrating factor for
\[ M(x, y)\, dx + N(x, y)\, dy = 0 \]
if and only if it is a solution to the partial differential equation
\[ N \frac{\partial I}{\partial x} - M \frac{\partial I}{\partial y} = \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) I. \]

Theorem 1.9.4. Consider the differential equation $M(x, y)\, dx + N(x, y)\, dy = 0$.

1. There exist an integrating factor that is dependent only on $x$ if and only if $\frac{M_y - N_x}{N} = f(x)$ which is a function of $x$ only. In this case,
\[ I(x) = e^{\int f(x)\, dx}. \]

2. There exist an integrating factor that is dependent only on $y$ if and only if $\frac{M_y - N_x}{M} = g(y)$ which is a function of $y$ only. In this case,
\[ I(y) = e^{-\int g(y)\, dy}. \]

Example 19. Find the general solution to $(xy - 1)\, dx + x^2\, dy = 0$
1.11 Some Higher-Order Differential Equations

**Note.** We will now look at how to solve particular second-order differential equations by replacing them with an equivalent first-order differential equation.

**Second-Order Differential Equation:** \[ \frac{d^2 y}{dx^2} = F\left(x, y, \frac{dy}{dx}\right) \]

**Second-Order Equations with the Dependent Variable Missing:** If \( y \) does not occur explicitly in the function \( F \), then

**Second-Order Equations with the Independent Variable Missing:** If \( x \) does not occur explicitly in the function \( F \), then
Example 20. Find the general solution to $y'' - 2x^{-1}y' = 18x^4$. 
**Example 21.** Find the general solution to $y'' + y^{-1}(y')^2 = ye^{-y(y')^3}$. 
2 Matrices and Systems of Linear Equations

2.1 Matrices: Definition and Notation

**Definition.** An $m \times n$ **matrix** is a rectangular array of numbers arranged in $m$ horizontal rows and $n$ vertical columns. Matrices are usually denoted by uppercase letters, such as $A$ and $B$. The entries in the matrix are called the **elements** of the matrix.

A general $m \times n$ matrix $A$ is written as

\[
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn}
\end{bmatrix},
\]

or equivalently, $A = [a_{ij}]$.

**Definition.** Two matrices $A$ and $B$ are **equal**, written $A = B$, if

1. They both have the same size, $m \times n$.
2. All corresponding elements in the matrices are equal: $a_{ij} = b_{ij} \forall i, j$ with $1 \leq i \leq m$ and $1 \leq j \leq n$.

**Definition.** A $1 \times n$ matrix is called a **row $n$-vector**. An $n \times 1$ is called a **column $n$-vector**. The elements of a row or column $n$-vector are called the **components** of the vector.

\[
\vec{a} = \begin{bmatrix} e & -0.2 & \frac{3}{7} & \sin 5 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} -42 \\ 5^{-3} \\ 0 \end{bmatrix}
\]
Example 1. Find the matrix and its size given \( \vec{b}_1 = [-4 \ 0] \), \( \vec{b}_2 = \begin{bmatrix} 13 \\ 5 \\ 2 \end{bmatrix} \), \( \vec{b}_3 = [0 \ \sqrt{3}] \), and \( \vec{b}_4 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 3 \end{bmatrix} \).

**Definition.** If we interchange the row vectors and column vectors in an \( m \times n \) matrix \( A \), we obtain an \( n \times m \) matrix called the transpose of \( A \). We denote this matrix by \( A^T \). This means that \( a_{ij} = a_{ji} \) for all \( i, j \) with \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \).

Example 2. Find \( A^T \) and its size for the above matrix. Also, write the matrix as a matrix of column vectors.

**Definition.** An \( n \times n \) matrix is called a square matrix, since it has the same number of rows as columns.

**Definition.** If \( A \) is a square matrix, then the elements \( a_{ii}, 1 \leq i \leq n \), make up the main diagonal or leading diagonal.

**Definition.** The sum of the main diagonal elements of an \( n \times n \) matrix \( A \) is called the trace of \( A \) and is denoted \( tr(A) \). Thus,

\[
tr(A) = a_{11} + a_{22} + \cdots + a_{nn}.
\]
**Example 3.** For the given matrix, determine its size, its leading diagonal, and its trace.

\[
A = \begin{bmatrix}
-3 & 4 & 6 \\
5 & -\frac{1}{2} & 4 \\
0 & 0 & 7/2 \\
\end{bmatrix}
\]

---

**Definition.** An \(n \times n\) matrix \(A\) is said to be **lower triangular** if \(a_{ij} = 0\) whenever \(i < j\) and it is said to be **upper triangular** if \(a_{ij} = 0\) whenever \(i > j\).

If \(a_{ij} = 1\) whenever \(i = j\) (main diagonal) of a lower(upper) triangular matrix, the matrix is called **unit lower** (upper) **triangular matrix**.

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**Definition.** An \(n \times n\) matrix \(D = [d_{ij}]\) is said to be **diagonal** when every off-diagonal element is a zero, \(d_{ij} = 0\) when \(i \neq j\). This means that \(D\) is both upper and lower triangular. We can therefore denote \(D\) in the compact form \(D = diag(d_1, d_2, \ldots, d_n)\).

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**Example 4.**

(a) Example of a \(3 \times 3\) unit upper triangular matrix.

(b) Write the matrix \(D = diag(1, \sqrt{2}, \sqrt{3}, 2)\) in its expanded form.
**Definition.** Given a square $n \times n$ matrix $A$,

1. If $A^T = A$ is called a **symmetric matrix**.

2. If $A = [a_{ij}]$, then we let $-A$ denote the matrix with elements $-a_{ij}$. A square matrix $A$ satisfying $A^T = -A$ is called a **skew-symmetric** (or **anti-symmetric**) matrix.

**Definition.** An $m \times n$ matrix function $A$ is a rectangular array with $m$ rows and $n$ columns whose elements are functions of a single real variable $t$.

**Definition.** An $n \times 1$ matrix function is called a **column** $n$-vector function.

**Example 5.** Determine the values $t$ for which $A(t) = \begin{bmatrix} t & e^t \\ \ln(5 - t) & (t + 2)^{-1/2} \\ \sin t & -3 \end{bmatrix}$ is defined.
2.2 Matrix Algebra

**Definition.** Let $k, h \in \mathbb{C}$ and $A, B$ and $C$ be matrices.

1. **Scalar Multiplication** $kA$: Each element of $A$ is multiplied by $k$. So if $A = [a_{ij}]$, then $kA = [ka_{ij}]$.
   
   a) $1A = A$  
   b) $k(hA) = (kh)A$  
   c) $(k + h)A = kA + hA$  
   d) $k(A + B) = kA + kB$

2. **Matrix Addition and Subtraction**, $A \pm B$: add/subtract each $i, j$-element of $A$ with the $i, j$-element of $B$. So if $A = [a_{ij}]$ and $B = [b_{ij}]$, then $A \pm B = [a_{ij} \pm b_{ij}]$.
   
   a) Matrices must be the same size  
   b) $A + B = B + A$  
   c) $(A + B) + C = A + (B + C)$

**Example 6.** Find the following:

(a) $(i - 2) \begin{bmatrix} 4 & 5 \\ -6i & 7 \\ 1 & 2i \end{bmatrix}$  
(b) $6\sqrt{2} \begin{bmatrix} 3 & 0 & 1 \\ -7 & 8 & 9 \end{bmatrix} - 4\sqrt{2} \begin{bmatrix} -3 & 10 & -2 \\ 3 & 5 & 1 \end{bmatrix}$

**Definition.** The **zero matrix**, $0_{m \times n}$, is the $m \times n$ matrix with all zero elements. In the case that the matrix is square of size $n \times n$, we denote the matrix as $0_n$.

**Properties:**

1. $A_{mn} + 0_{mn} = A_{mn}$  
2. $A_{mn} - A_{mn} = 0_{mn}$  
3. $0A_{mn} = 0_{mn}$
Matrix Multiplication, \( AB \): the number of columns of \( A \) must equal the number of rows of \( B \), i.e. \( A = m \times n \) and \( B = n \times p \) and the resulting matrix will be of size \( m \times p \). There are three cases we will cover.

1. Product of row \( n \)-vector and a column \( n \)-vector: (think “dot product”). Let \( \vec{a} \) be a row \( n \)-vector and \( \vec{x} \) be a column \( n \)-vector. Then \( \vec{a} \) is of size \( 1 \times n \) and \( \vec{x} \) is of size \( n \times 1 \). The resulting product will be a \( 1 \times 1 \) matrix whose element is the dot product of the row vectors of \( \vec{a} \) and \( \vec{x}^T \). So,

\[
\vec{a} \cdot \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.
\]

2. Product of an \( m \times n \) matrix with a column \( n \)-vector: Let \( A \) be an \( m \times n \) matrix and \( \vec{x} \) be a column \( n \)-vector. Then \( A \vec{x} \) is an \( m \times 1 \) matrix whose \( i \)th element is obtained by taking the dot product of the \( i \)th row vector of \( A \) with \( \vec{x}^T \). So,

\[
A \vec{x} = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \\
a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \\
\vdots \\
a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n
\end{bmatrix} = \begin{bmatrix} (A \vec{x})_1 \\
( A \vec{x})_2 \\
\vdots \\
( A \vec{x})_n
\end{bmatrix}
\]

where \((A \vec{x})_i = \sum_{k=1}^n a_{ik} x_k\) for \(1 \leq i \leq m\).

**Theorem 2.2.1.** If \( A = [\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n] \) is an \( m \times n \) matrix and \( \vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \) is column \( n \)-vector, then

\[
A \vec{c} = c_1 \vec{a}_1 + c_2 \vec{a}_2 + \cdots + c_n \vec{a}_n.
\]

**Proof:**
**Definition.** If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are column $m$-vectors and $c_1, c_2, \ldots, c_n$ are scalars, then an expression of the form $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$ is a **linear combination** of the column vectors.

**Example 7.** Find \[
\begin{bmatrix}
1 & 2 & -3 \\
0 & 6 & -4 \\
-5 & 7 & 8
\end{bmatrix}
\begin{bmatrix}
3 \\
2 \\
-1
\end{bmatrix}.
\]

3. **Product of an $m \times n$ and an $n \times p$ matrix:** Think “rows of $A$ times columns of $B$.” If $A$ is an $m \times n$ matrix and $B$ is an $n \times p$ matrix where $B = [\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p]$ then $AB$ is an $m \times p$ matrix defined by $AB = [A\mathbf{b}_1, A\mathbf{b}_2, \ldots, A\mathbf{b}_p]$.

**Example 8.** Find \[
\begin{bmatrix}
4 & -2 & 3 \\
0 & 1 & 2 \\
-1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
2 & 6 \\
4 & 0 \\
-3 & 5
\end{bmatrix}.
\]
**Definition.** If $A = [a_{ij}]$ is an $m \times n$ matrix, $B = [b_{ij}]$ is an $n \times p$ matrix, and $C = AB$, then

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}, \quad 1 \leq i \leq m, \quad 1 \leq j \leq p.$$  

This is called the **index form** of the matrix product.

**Theorem 2.2.2.** If $A$, $B$, and $C$ have appropriate dimensions for the operations to be performed, then

1. $AB \neq BA$
2. $A(BC) = (AB)C$
3. $A(B + C) = AB + AC$
4. $(A + B)C = AC + BC$

**Proof:**

**Definition.** The elements of $I_n$ can be represented by the **Kronecker delta symbol**, $\delta_{ij}$, defined by

$$\delta_{ij} = \begin{cases} 
1, & \text{if } i = j \\
0, & \text{if } i \neq j.
\end{cases}$$

Then,

$$I_n = [\delta_{ij}].$$

**Identity Matrix Properties:**

1. $A_{m\times n}I_n = A_{m\times n}$
2. $I_mA_{m\times n} = A_{m\times n}$

**Proof:**
Theorem 2.2.3. Let $A$ and $C$ be $m \times n$ matrices, and let $B$ be an $n \times p$ matrix. Then

1. $(A^T)^T = A$
2. $(A + C)^T = A^T + C^T$
3. $(AB)^T = B^T A^T$

Proof:

Theorem 2.2.4. The product of two lower (upper) triangular matrices is a lower (upper) triangular matrix.

Proof:
2.3 Terminology for Systems of Linear Equations

**Definition.** The general $m \times n$ system of linear equations is of the form

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\
    \vdots \\
    a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{align*}
\]

where the **system coefficients** $a_{ij}$ and the **system constant** $b_j$ are given scalars and $x_1, x_2, \ldots, x_n$ denote the unknowns in the system.

**Definition.** If $b_i = 0 \ \forall i$, then the system is called **homogeneous**; otherwise it is called **nonhomogeneous**.

**Definition.** By a **solution** to the system above we mean an ordered $n$-tuple of scalars, $(c_1, c_2, \ldots, c_n)$, which, when substituted for $x_1, x_2, \ldots, x_n$ into the left-hand side of the system above, yield the values on the right-hand side. The set of all solutions to the system above is called the **solution set** to the system.

**Definition.** A system of equations that has at least one solution is said to be **consistent**, whereas a system that has no solutions is called **inconsistent**.

**Definition.** Given the general $m \times n$ system of linear equations:

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\
    \vdots \\
    a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{align*}
\]

**Matrix of Coefficients** $A$  
**Augmenter Matrix** $A^#$
Definition. The general $m \times n$ system of linear equations above can be written as $A\vec{x} = \vec{b}$ is the matrix coefficients and

$$
\vec{x} = \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix}
b_1 \\
b_2 \\
\vdots \\
b_m
\end{bmatrix}
$$

where $\vec{x}$ is the vector of unknowns, $\vec{x} \in \mathbb{R}^n$, and $\vec{b}$ is the right-hand-side, $\vec{b} \in \mathbb{R}^m$. Then the solution set to the vector equation $A\vec{x} = \vec{b}$ is

$$
S = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{b}\}.
$$
2.4 Elementary Row Operations and Row-Echelon Matrices

**Note.** We will not be using the same notation for Elementary Row Operations as the book. However, it is important to realize that each elementary row operation is reversible; we can “undo” a given elementary row operation by another elementary row operation to bring the modified linear system back into its original form.

**Definition.** Let $A$ be an $m \times n$ matrix. Any matrix obtained from $A$ by a finite sequence of elementary row operations is said to be **row-equivalent** to $A$.

**Theorem 2.4.1.** Systems of linear equations with row-equivalent augmented matrices have the same solutions sets.

**Definition.** An $m \times n$ matrix is called **row-echelon matrix** if it satisfies the following three conditions:

1. If there are any rows consisting entirely of zeros, they are grouped together at the bottom of the matrix.
2. The first nonzero element in any nonzero row (column), any row(column) that does not consist entirely of zeros, is a 1 (called a **leading 1**).
3. The leading 1 of any row below the first row is to the right of the leading 1 of the row above it.

**Example 9.** Use elementary row operations to reduce $A$ to row-echelon form: $A = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & -1 & 2 & 1 \\ -4 & 6 & -7 & 1 \\ 2 & 0 & 1 & 3 \end{bmatrix}$

**Theorem 2.4.2.** Any matrix is row-equivalent to a row-echelon matrix.
Note. When solving a system, once one value is found, substituting it in to an equation that contains that variable and one other variable to find the other variable and so on to finish solving the entire system. This method is called back substitution.

Note. See page 145 on your book to see Algorithm for Reducing an $m \times n$ Matrix $A$ to Row-Echelon Form.

Theorem 2.4.3. Let $A$ be an $m \times n$ matrix. All row-echelon matrices that are row-equivalent to $A$ have the same number of nonzero rows.

Definition. The number of nonzero rows in any row-echelon form of a matrix $A$ is called the rank of $A$ and is denoted $\text{rank}(A)$.

Example 10. Find the rank of Example 1.

Example 11. Reduce $A = \begin{bmatrix} 0 & 2 & 8 & -7 \\ 2 & -2 & 4 & 0 \\ -3 & 4 & -2 & -5 \end{bmatrix}$ to row-echelon form and find its rank.
Example 12. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 0 & 1 \end{bmatrix}$. We can write $A$ as $A = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \end{bmatrix}$. Matrix $A$ reduces to $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. What is its rank?

Note. If $A$ is an $m \times n$ matrix, then $\text{rank}(A) \leq m$ and $\text{rank}(A) \leq n$.

Definition. An $m \times n$ matrix is called a reduced row-echelon matrix if it satisfies the following condition:

1. It is a row-echelon matrix.
2. Any column that contains a leading 1 has zeros everywhere else.

Example 13. Determine the reduced row-echelon form and rank of $A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$.
2.5 Gaussian Elimination

**Definition.** The process of reducing the augmented matrix to row-echelon form and then using back substitution to solve the equivalent system is called *Gaussian elimination*.

The case of Gaussian elimination when the augmented matrix is reduced to reduced row-echelon form is called *Gauss-Jordan elimination*.

**Example 14.** Use Gaussian elimination to solve the system of linear equations:

\[
\begin{align*}
2x_2 + x_3 &= -8 \\
x_1 - 2x_2 - 3x_3 &= 0 \\
-x_1 + x_2 + 2x_3 &= 3
\end{align*}
\]

**Lemma 2.5.1.** Consider the \(m \times n\) linear system \(A\vec{x} = \vec{b}\). Let \(A^\#\) denote the augmented matrix of the system. If \(\text{rank}(A) = \text{rank}(A^\#) = n\), then the system has a unique solution.

**Proof:**
Example 15. Use Gaussian elimination to solve the system of linear equations:

\[
\begin{align*}
  x_1 - 2x_2 - 6x_3 &= 12 \\
  2x_1 + 4x_2 + 12x_3 &= -17 \\
  x_1 - 4x_2 - 12x_3 &= 22
\end{align*}
\]

Lemma 2.5.2. Consider the \( m \times n \) linear system \( A\tilde{x} = \tilde{b} \). Let \( A^\# \) denote the augmented matrix of the system. If \( \text{rank}(A) < \text{rank}(A^\#) \), then the system is inconsistent.

Proof:

Definition. The variable that we choose to specify when a solution has an infinite number of solutions is called a free variable or free parameter.

The remaining variables are then determined by the system of equations, in terms of the free variable(s)/free parameter(s), and are called bound variables or bound parameters.
Consider the matrix

\[
A = \begin{bmatrix}
1 & -2 & -6 & 12 & 1 \\
2 & 4 & 12 & -17 & 2 \\
2 & 4 & 12 & -17 & 2
\end{bmatrix}
\]

. Matrix A reduces to

\[
\begin{bmatrix}
1 & -2 & 6 & -\frac{17}{2} & 1 \\
0 & 1 & 3 & -\frac{41}{8} & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Lemma 2.5.3. Consider the \( m \times n \) linear system \( A\vec{x} = \vec{b} \). Let \( A^# \) denote the augmented matrix of the system and let \( r^# = \text{rank}(A^#) \). If \( r^# = \text{rank}(A) < n \), then the system has an infinite number of solutions, indexed by \( n - r^# \) free variables.

Proof:

Note. Choose as the free variables those that DO NOT correspond to a leading 1 in a row-echelon form of the augmented matrix.
Theorem 2.5.1. Consider the $m \times n$ linear system $A\vec{x} = \vec{b}$. Let $r$ denote the rank of $A$, and let $r^\#$ denote the rank of the augmented matrix of the system. Then

1. If $r < r^\#$, the system is inconsistent.

2. If $r = r^\#$, the system is consistent and
   a) There exists a unique solution $\iff r^\# = n$.
   b) There exists an infinite number of solutions $\iff r^\# < n$.

Corollary 2.5.1. The homogeneous linear system $A\vec{x} = \vec{0}$ is consistent for any coefficient matrix $A$, with a solution given by $\vec{x} = \vec{0}$.

Corollary 2.5.2. A homogeneous system of $m$ linear equations in $n$ unknowns, with $m < n$, has an infinite number of solutions.

Example 16. Determine the solution set to the system $A\vec{x} = \vec{0}$, if $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{bmatrix}$.
2.6 The Inverse of a Square Matrix

**Theorem 2.6.1.** Let \( A \) be an \( n \times n \) matrix. Suppose \( B \) and \( C \) are both \( n \times n \) matrices satisfying \( AB = BA = I_n \) and \( AC = CA = I_n \) respectively. Then \( B = C \).

**Proof:**

**Definition.** Let \( A \) be an \( n \times n \) matrix. If there exist and \( n \times n \) matrix \( A^{-1} \) satisfying \( AA^{-1} = A^{-1}A = I_n \), then we call \( A^{-1} \) the matrix inverse to \( A \), or just the inverse of \( A \). We say that \( A \) is invertible if \( A^{-1} \) exist.

**Note.** Invertible matrices are sometimes called nonsingular, while matrices that are not invertible are sometimes called singular.

**Corollary 2.6.1.** Let \( A \) be an \( n \times n \) matrix. If \( A\tilde{x} = \tilde{b} \) has a unique solution for some column \( n \)-vector \( \tilde{b} \), then \( A^{-1} \) exists.

**Gauss-Jordan Technique for Finding the Inverse of a Nonsingular (Invertible) Matrix:**

1. Form the matrix \([A | I_n]\).
2. Transform the matrix into reduced row-echelon form.
3. The reduced row-echelon form contains the identity matrix on the left and the inverse of \( A \) on the right, i.e. \([I_n \mid A^{-1}]\).
Example 17. Find the inverse of the matrix using the Gauss-Jordan method. \( A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \).

Example 18. Find the inverse of the matrix using Gauss-Jordan method. \( A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \).
**Theorem 2.6.2.** If $A^{-1}$ exists, then the $n \times n$ system of linear equations

$$A\tilde{x} = \tilde{b}$$

has the unique solution

$$\tilde{x} = A^{-1}\tilde{b}, \forall \tilde{b} \in \mathbb{R}^n.$$ 

**Example 19.** Find the solution to the given system

\[
\begin{align*}
    x_1 + 3x_2 &= 1 \\
    2x_1 + 5x_2 &= 3
\end{align*}
\]

**Theorem 2.6.3.** An $n \times n$ matrix $A$ is invertible $\iff$ rank$(A) = n.$
Theorem 2.6.4. Let $A$ and $B$ be invertible $n \times n$ matrices. Then

1. $A^{-1}$ is invertible and $(A^{-1})^{-1} = A$.

2. $AB$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

3. $A^T$ is invertible and $(A^T)^{-1} = (A^{-1})^T$.

Proof:

Corollary 2.6.2. Let $A_1, A_2, \ldots, A_k$ be invertible $n \times n$ matrices. Then $A_1A_2\cdots A_k$ is invertible, and

$$(A_1A_2\cdots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1}$$

Theorem 2.6.5. Let $A$ and $B$ be $n \times n$ matrices. If $AB = I_n$, then both $A$ and $B$ are invertible and $B = A^{-1}$.

Corollary 2.6.3. Let $A$ and $B$ be $n \times n$ matrices. If $AB$ is invertible, then both $A$ and $B$ are invertible.
3 Determinants

3.1 The Definition of the Determinant

**Note.** From §2.6 that “An $n \times n$ matrix $A$ is invertible $\iff$ $\text{rank}(A) = n$.”

Let $A$ be $n \times n$ matrix. We will be looking at three different scenarios, when $n = 1, 2$ and 3.

**Scenario 1:**

**Scenario 2:**

**Scenario 3:**

**Definition.** Consider the first $n$ positive integers $1, 2, 3, \ldots, n$. Any arrangement of these integers in a specific order, say, $(p_1, p_2, \ldots, p_n)$, is called a **permutation**.

**Example 1.** Given the integers 1, 2, and 3, what are and how many distinct permutations are there of these three numbers?

**Theorem 3.1.1.** There are precisely $n!$ distinct permutations of the integers $1, 2, 3, \ldots, n$. 

**Definition.** For \( i \neq j \), the pair of elements \( p_i \) and \( p_j \) in the permutation \((p_1, p_2, \ldots, p_n)\) are said to be inverted if they are out of their natural order; that is, if \( p_i > p_j \) with \( i < j \). If this is the case, we say that \((p_i, p_j)\) is an inversion.

**Example 2.** Find the number of inversion for each permutation in Ex1.

**Definition.** The number of inversions enables us to distinguish two different types of permutations.

1. If \( N(p_1, p_2, \ldots, p_n) \) is an even integer (or zero), we say \((p_1, p_2, \ldots, p_n)\) is an even permutation. We also say that \((p_1, p_2, \ldots, p_n)\) have even parity.

2. If \( N(p_1, p_2, \ldots, p_n) \) is an odd integer, we say \((p_1, p_2, \ldots, p_n)\) is an odd permutation. We also say that \((p_1, p_2, \ldots, p_n)\) have odd parity.

**Example 3.** Find the parity of Ex1.

**Note.** A plus or a minus sign is associated with the permutation, depending on whether it has an even or odd parity, respectively.

\[
\sigma(p_1, p_2, \ldots, p_n) = \begin{cases} 
+1 & \text{if } \sigma(p_1, p_2, \ldots, p_n) \text{ has even parity}, \\
-1 & \text{if } \sigma(p_1, p_2, \ldots, p_n) \text{ has odd parity}.
\end{cases}
\]

Hence,

\[
\sigma(p_1, p_2, \ldots, p_n) = (-1)^N(p_1, p_2, \ldots, p_n).
\]

**Theorem 3.1.2.** If any two elements in a permutation are interchanged, then the parity of the resulting permutation is opposite to that of the original permutation.
**Definition.** Let $A = [a_{ij}]$ be an $n \times n$ matrix. The determinant of $A$, denoted $\det(A)$ or $|A|$, is defined as follows:

$$\det(A) = |A| = \sum \sigma(p_1, p_2, \ldots, p_n)a_{1p_1}a_{2p_2}a_{3p_3} \cdots a_{np_n},$$

where the summation is over the $n!$ distinct permutation $(p_1, p_2, \ldots, p_n)$ of the integers $1, 2, 3, \ldots, n$. The determinant of an $n \times n$ matrix is said to have order $n$.

**Example 4.** Let’s derive the determinant of a $3 \times 3$ matrix.

**Note.** There are other ways to find the determinant of a $3 \times 3$ matrix.
Example 5. Find the determinant of the following matrices

\[ A = \begin{bmatrix} \pi & \pi^2 \\ \sqrt{2} & 2\pi \end{bmatrix} \quad B = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & -3 \\ 4 & -1 & 5 \end{bmatrix} \]

**Note.** Read pg 194 - 195 for a Geometric Interpretation of the Determinants of Order Two and Three. *If you try to go any higher, you may get a headache.*
3.2 Properties of Determinants

**Theorem 3.2.1.** If $A$ is an $n \times n$ upper or lower triangular matrix, then

$$\det(A) = a_{11}a_{22}a_{33} \cdots a_{nn} = \prod_{i=1}^{n} a_{ii}. $$

**Proof:**

---

**Properties of Determinants**

Let $A$ be an $n \times n$ matrix.

1. If $B$ is the matrix obtained by permuting two rows of $A$, then $\det(B) = -\det(A)$.

2. If $B$ is the matrix obtained by multiplying one row of $A$ by any scalar $k$, then $\det(B) = k\det(A)$.

3. If $B$ is the matrix obtained by adding a multiple of any row of $A$ to a different row of $A$, then $\det(B) = \det(A)$.

**Proof:**
Example 6. Find $\det(B)$ for $B = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$

Theorem 3.2.2. Let $A$ be an $n \times n$ matrix with real elements. The following conditions on $A$ are equivalent:

1. $A$ is invertible.
2. $\det(A) \neq 0$.

Example 7. Find the $A^{-1}$ of $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 5 & -7 \\ -3 & -6 & 9 \end{bmatrix}$ if it exist.
Corollary 3.2.1. The homogeneous $n \times n$ linear system $A\vec{x} = \vec{0}$ has an infinite number of solutions if and only if $\det(A) = 0$, and has only the trivial solution if and only if $\det(A) \neq 0$.

More Properties of Determinants

Let $A$ be $B$ an $n \times n$ matrix.

4. $\det(A^T) = \det(A)$.

5. Let $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ denote the row vectors of $A$. If the $i$th row vector $A$ is the sum of two vectors, say $\vec{a}_i = \vec{b}_i + \vec{c}_i$, then $\det(A) = \det(B) + \det(C)$, where

$$B = \begin{bmatrix} \vec{b}_1 \\ \vdots \\ \vec{b}_{i-1} \\ \vec{b}_i \\ \vec{b}_{i+1} \\ \vdots \\ \vec{b}_n \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} \vec{c}_1 \\ \vdots \\ \vec{c}_{i-1} \\ \vec{c}_i \\ \vec{c}_{i+1} \\ \vdots \\ \vec{c}_n \end{bmatrix}$$

The corresponding property is also true for columns.

6. If $A$ has a row (or column) of zeros, then $\det(A) = 0$.

7. If two rows (or columns) of $A$ are the same, then $\det(A) = 0$.

8. $\det(AB) = \det(A)\det(B)$. 
Example 8. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and assume $\text{det}(A) = -6$. Find the $\text{det}(B)$ given:

$$B = \begin{bmatrix} d & e & f \\ -3a & -3b & -3c \\ g - 4d & h - 4e & i - 4f \end{bmatrix}.$$
3.3 Cofactor Expansion

**Definition.** Let $A$ be an $n \times n$ matrix.

1. The minor, $M_{ij}$, of the element $a_{ij}$, is the determinant of the matrix obtained by deleting the $i$th row vector and $j$th column vector of $A$.

2. The cofactor, $C_{ij}$, of the element $a_{ij}$, is defined by

$$C_{ij} = (-1)^{i+j}M_{ij},$$

where $M_{ij}$ is the minor of $a_{ij}$.

Hence, the **Sign of Cofactors** matrix is

$$
\begin{vmatrix}
+ & - & + & - & + & \cdots \\
- & + & - & + & - & \cdots \\
+ & - & + & - & + & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{vmatrix}
$$

**Example 9.** Find the cofactors of $a_{22}$ and $a_{23}$ given $A = \begin{bmatrix} -3 & 2 & 8 \\ 4 & 1 & -5 \\ 6 & 7 & 2 \end{bmatrix}$.

**Theorem 3.3.1.** Let $A$ be an $n \times n$ matrix. If we multiply the elements in any row (or column) of $A$ by their cofactors, then the sum of the resulting products is $\det(A)$. Thus,

1. If we expand along row $i$,

$$
\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{k=1}^{n} a_{ik}C_{ik}.
$$

2. If we expand along column $j$,

$$
\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{k=1}^{n} a_{kj}C_{kj}.
$$
Example 10. Compute the determinant of \( A = \begin{bmatrix} 2 & -3 & 1 & 0 \\ 4 & 3 & 2 & 1 \\ 1 & -4 & 5 & 0 \\ 3 & 2 & 6 & -7 \end{bmatrix} \) using cofactor expansion.

Corollary 3.3.1. If the elements in the \( i \)th row (or column) of an \( n \times n \) matrix \( A \) are multiplied by the cofactors of a different row (or column), then the sum of the resulting products is zero. That is,

1. If we use the elements of row \( i \) and the cofactors of row \( j \),

\[
\sum_{k=1}^{n} a_{ik} C_{jk} = 0, \quad i \neq j.
\]

2. If we use the elements of column \( i \) and the cofactors of column \( j \),

\[
\sum_{k=1}^{n} a_{ki} C_{kj} = 0, \quad i \neq j.
\]
Corollary 3.3.2. Let $A$ be an $n \times n$ matrix. If $\delta_{ij}$ is the Kronecker delta symbol then,

$$\sum_{k=1}^{n} a_{ik}C_{jk} = \delta_{ij} \text{det}(A), \quad \sum_{k=1}^{n} a_{ki}C_{kj} = \delta_{ij} \text{det}(A)$$

**Definition.** If every element in an $n \times n$ matrix $A$ is replaced by its cofactor, the resulting matrix is called the **matrix of cofactors** and is denoted $M_C$. The transpose of the matrix of cofactors, $M_C^T$, is called the **adjoint** of $A$ and is denoted $\text{adj}(A)$. Thus, the elements of $\text{adj}(A)$ are

$$\text{adj}(A)_{ij} = C_{ji}.$$  

**Theorem 3.3.2.** *(The Adjoint Method of Computing $A^{-1}$)*

If $\text{det}(A) \neq 0$, then

$$A^{-1} = \frac{1}{\text{det}(A)} \text{adj}(A).$$

**Example 11.** Given matrix $A$, if possible, find $A^{-1}$.

$$A = \begin{bmatrix} 5 & 5 & -2 \\ 0 & -1 & 2 \\ 4 & 3 & 6 \end{bmatrix}.$$
Theorem 3.3.3. If $\det(A) \neq 0$, the unique solution to the $n \times n$ system $A\mathbf{x} = \mathbf{b}$ is $(x_1, x_2, \ldots, x_n)$, where

$$x_k = \frac{\det(B_k)}{\det(A)}, \quad k = 1, 2, \ldots, n.$$ 

Example 12. Use Cramer’s Rule to solve the following system:

$$\begin{align*}
2x_1 + 3x_2 + x_3 &= 10 \\
x_1 - x_2 + x_3 &= 4 \\
4x_1 - x_2 - 5x_3 &= -8.
\end{align*}$$
4 Vector Spaces

4.2 Definition of a Vector Space

**Definition.** A **field**, \( F \), is a place where all arithmetic can be performed. It has two operations, an addition (+) and a multiplication (× or ⋅). The operations have the following properties:

I. **Commutativity:** \( \forall a, b \in F, \ a + b = b + a \) and \( ab = ba \).

II. **Associativity:** \( \forall a, b, c \in F, \ a + (b + c) = (a + b) + c \) and \( a(bc) = (ab)c \).

III. **Existence of unique identities:** respectively known as 0 and 1: \( \forall a \in F, \ a + 0 = 0 + a \) and \( a \times 1 = 1 \times a \) where \( 0 \neq 1 \).

IV. **Existence of inverses:** \( \forall a, b \in F, \ b \neq 0, \ \exists \ -a, b^{-1} \in F \) such that \( a + (-a) = 0 = (-a) + a \) and \( b \times b^{-1} = 1 = b^{-1} \times b \) where \( -a \) is called the “negative of \( a \)” or “minus \( a \)” and \( b^{-1} \) is called the “the inverse of \( b \).” This leads us to two more operations: subtraction and division.

V. **Distributivity:** \( \forall a, b, c \in F, \ a(b + c) = ab + ac \).

**Note.** The two fields we will be using in the class are \( \mathbb{R} \), or the set of real numbers, and \( \mathbb{C} \), which is the set of complex numbers.

**Definition.** Let \( V \) be a nonempty set (whose elements are called vectors) on which are defined an addition and a scalar multiplication operation with scalars in \( F \). We call \( V \) a **vector space over** \( F \), provided the following ten condition are satisfied:

1. **Closure under addition:** \( \forall \vec{u}, \vec{v} \in V, \ \vec{u} + \vec{v} \in V \).
2. **Closure under scalar multiplication:** \( \forall a \in F \) and \( \forall \vec{u} \in V, \ a\vec{u} \in V \).
3. **Commutativity of addition:** \( \forall \vec{u}, \vec{v} \in V, \ \vec{u} + \vec{v} = \vec{v} + \vec{u} \).
4. **Associativity of addition:** \( \forall \vec{u}, \vec{v}, \vec{w} \in V, \ (\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}) \).
5. **Existence of a zero vector in** \( V \): \( \exists \vec{0} \in V \), satisfying \( \vec{v} + \vec{0} = \vec{v} \ \forall \vec{v} \in V \).
6. **Existence of additive inverses in** \( V \): \( \forall \vec{u} \in V, \exists -\vec{u} \in V \) satisfying \( \vec{u} + (-\vec{u}) = \vec{0} \).
7. **Unit property:** \( \forall \vec{u} \in V, \ 1\vec{u} = \vec{u} \).
8. **Associativity of scalar multiplication:** \( \forall \vec{u} \in V \) and \( \forall a, b \in F, \ (ab)\vec{u} = a(b\vec{u}) \).
9. **Distributive property of scalar multiplication over vector addition:** \( \forall \vec{u}, \vec{v} \in V, \) and \( \forall a \in F, \ a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v} \).
10. **Distributive property of scalar multiplication over scalar addition:** \( \forall \vec{u} \in V, \) and \( \forall a, b \in F, \ (a + b)\vec{u} = a\vec{u} + b\vec{u} \).
Note. We will call the vector space over the real numbers the real vector space and the vector space over the complex numbers the complex vector space.

Example 1. The following are examples of vector spaces.
Important Vector Spaces (for this course)

- \( \mathbb{R}^n \), the real vector space of all ordered \( n \)-tuples of real numbers.
- \( \mathbb{C}^n \), the complex vector space of all ordered \( n \)-tuples of complex numbers.
- \( M_{m \times n}(\mathbb{R}) \), the real vector space of all \( m \times n \) matrices with real elements.
- \( M_n(\mathbb{R}) \), the real vector space of all \( n \times n \) matrices with real elements.
- \( C^k(I) \), the vector space of all real-valued functions that are continuous and have (at least) \( k \) continuous derivatives on \( I \).
- \( P_n \), the real vector space of all real-valued polynomials of degree \( \leq n \) with real coefficients. That is,
  \[
  P_n = \{ a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n : a_0, a_1, \ldots, a_n \in \mathbb{R} \}.
  \]

**Theorem 4.2.1.** Let \( V \) be a vector space over \( F \).

1. The zero vector is unique.
2. \( 0 \vec{u} = \vec{0} \) \( \forall \vec{u} \in V \).
3. \( k \vec{0} = \vec{0} \) \( \forall k \in F \).
4. The additive inverse of each element in \( V \) is unique.
5. \( \forall \vec{u} \in V, -\vec{u} = (-1)\vec{u} \).
6. If \( k \) is a scalar and \( \vec{u} \in V \) such that \( k\vec{u} = \vec{0} \), then either \( k = 0 \) or \( \vec{u} = \vec{0} \).

**Proof:**
4.3 Subspaces

Example 2. Consider the $P_2$ vector space. Now, let’s go back to calculus and think of a differential equation, say $\frac{dy}{dx} = 2x$. What would be its solution?

This means that the solutions to this differential equation are elements, or vectors, of $P_2$. Now the question is: Do the solutions to this particular differential equation form a vector space of their own? This is the motivation for the next section.

**Definition.** Let $S$ be a nonempty subset of a vector space $V$. If $S$ is itself a vector space under the same operations of addition and scalar multiplication as used in $V$, then we say that $S$ is a subspace of $V$.

**Theorem 4.3.1.** Let $S$ be a nonempty subset of a vector space $V$. Then $S$ is a subspace of $V$ $\iff$ $S$ is closed under the operations of addition and scalar multiplication in $V$.

Proof:
Note. If a subset $S$ of a vector space $V$ does not contain the vector $\vec{0}$, then it is not a subspace. Conversely, just because a subset contains the vector $\vec{0}$ does not guarantee that the subset is a subspace.

Example 3. Determine whether the set of $3 \times 3$ diagonal matrices form a subspace of $M_3(\mathbb{R})$.

Example 4. Determine whether the set solutions to $4x - 3y = 0$ form a subspace of $\mathbb{R}^2$. 
Example 5. Determine whether the set solutions to $4x - 3y = 1$ form a subspace of $\mathbb{R}^2$.

Theorem 4.3.2. Let $V$ be a vector space with zero vector $\vec{0}$. Then $S = \{\vec{0}\}$ is a subspace of $V$.

Proof:
Theorem 4.3.3. Let $A$ be an $m \times n$ matrix. The solution set of the homogeneous system of linear equations $A\vec{x} = \vec{0}$ is a subspace of $\mathbb{C}^n$.

Proof:

Note. The previous thm states that the solution set to any homogeneous linear system of equations is a vector space.

Definition. Let $A$ be an $m \times n$ matrix. The solution set to the corresponding homogeneous linear system $A\vec{x} = \vec{0}$ is called the null space of $A$ and is denoted $\text{nullspace}(A)$. Thus,

$$\text{nullspace}(A) = \{\vec{x} \in \mathbb{C}^n : A\vec{x} = \vec{0}\}.$$

1. If the matrix $A$ has real elements, then we will consider only the corresponding real solutions to $A\vec{x} = \vec{0}$. Consequently, in this case,

$$\text{nullspace}(A) = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\}.$$

a subspace of $\mathbb{R}^n$.

2. The previous theorem does not hold for the solution set of a nonhomogeneous linear system $A\vec{x} = \vec{b}$, for $\vec{b} \neq \vec{0}$, since $\vec{x} = \vec{0}$ is not in the solution set of the system.
4.4 Spanning Sets

**Definition.** Let \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in V \) and \( a_1, a_2, \ldots, a_k \in F \) where \( V \) is a vector space over a field \( F \), where \( a_i \) (\( i = 1, 2, \ldots, k \)) are scalars in \( F \). Any expression of the form \( a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k \) is called a **linear combination** of \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \).

**Definition.** If every vector in a vector space \( V \) can be written as a linear combination of \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \), we say that \( V \) is spanned or generated by \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \) and call the set of vectors \( \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \) a spanning set for \( V \). In this case, we also say that \( \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \) spans \( V \).

**Example 6.** Determine whether the set of given vectors span \( \mathbb{R}^2 \) : \{\((3, 2), (1, 1)\)\}.

**Theorem 4.4.1.** Let \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \) be vectors in \( \mathbb{R}^n \). Then \( \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \) spans \( \mathbb{R}^n \) if and only if, for the matrix \( A = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k] \), the linear system \( A\vec{c} = \vec{v} \) is consistent for every \( \vec{v} \in \mathbb{R}^n \).

**Proof:**

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Definition. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in V$, where $V$ is a vector space over a field $F$. All possible linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ generates a subset of $V$ called the linear span of $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$, denoted $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$.

Example 7. Let $\vec{v}_1 = (1, 2, 0)$ and $\vec{v}_2 = (3, 1, 1)$. Determine whether $\vec{w} = (4, -7, 3)$ lies in $\text{span}\{\vec{v}_1, \vec{v}_2\}$.

Theorem 4.4.2. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ be vectors in vector space $V$. Then $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is a subspace of $V$.

Proof:
Note. $\text{span}(\emptyset) = \{0\}$

**Example 8.** Determine whether the set of given vectors spans $\mathbb{R}^2$: $\{(1, 1), (-3, -3)\}$.

**Example 9.** Consider the set of vectors $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 4 & -3 \end{bmatrix}$ Determine the $\text{span}\{A, B\}$. 


Example 10. If \( p_1(x) = 3x + 4 \) and \( p_2(x) = 4x^2 - 1 \), determine the \( \text{span}\{p_1, p_2\} \) and whether \( p(x) = x^2 - x + 7 \) lies in this subspace.
4.5 Linear Dependence and linear independence

Notice that \{((1, -1, 1), (2, 5, 3), (4, -2, 1))\} and \{(2, 3, 1), (1, -2, 1), (4, -1, 2), (0, 0, 0)\} are all spanning sets for \(\mathbb{R}^3\)

**Definition.** A spanning set which contains the minimum number of vectors needed to span a given vector space is called a **minimal spanning set**.

**Theorem 4.5.1.** Let \(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}\) be a set of at least two vectors in a vector space \(V\). If one of the vectors in the set is a linear combination of the other vectors in the set, then that vector can be deleted from the given set of vectors and the linear span of the resulting set of vectors will be the same as the linear span of \(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}\).

**Proof:**

**Definition.** A finite nonempty set of vectors \(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}\) in a vector space \(V\) is said to be **linearly dependent** if there exist scalars \(a_1, a_2, \ldots, a_k \in F\), not all zero, such that

\[
a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k = \vec{0}.
\]

Such a nontrivial linear combination of vectors is sometimes referred to as a **linear dependency** among the vectors \(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\).

A finite, nonempty set of vectors \(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}\) in a vector space \(V\) is said to be **linearly independent** if the only values of the scalars \(a_1, a_2, \ldots, a_k \in F\) for which

\[
a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k = \vec{0}.
\]

are \(a_1 = a_2 = \cdots = a_k = 0\).
**Theorem 4.5.2.** A set consisting of a single vector $\vec{v}$ in a vector space $V$ is linearly dependent if and only if $\vec{v} = \vec{0}$. Therefore, any set of consisting of a single nonzero vector is linearly independent.

**Example 11.** Show that $\{\vec{v}\} = \vec{0}$ where $V$ is a vector space is
(a) linearly dependent when $\vec{v} = \vec{0}$  
(b) linearly independent when $\vec{v} \neq \vec{0}$.

**Theorem 4.5.3.** Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a set of at least two vectors in a vector space $V$. Then $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is linearly dependent if and only if at least one of the vectors in the set can be expressed as a linear combination of the others.

**Proposition 4.5.1.** Let $V$ be a vector space.
1. Any set of two vectors in $V$ is linearly dependent if and only if the vectors are proportional.
2. Any set of vectors in $V$ containing the zero vector is linearly dependent.

**Corollary 4.5.1.** Any nontrivial, finite set of linearly dependent vectors in a vector space $V$ contains a linearly independent subset that has the same linear span as the given set of vectors.

**Theorem 4.5.4.** Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ be vectors in $\mathbb{R}^n$ and $A = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k]$. Then $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is linearly dependent if and only if the linear system $A\vec{c} = \vec{0}$ has a nontrivial solution.

**Corollary 4.5.2.** Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ be vectors in $\mathbb{R}^n$ and $A = [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k]$.
1. If $k > n$, then $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is linearly dependent.
2. If $k = n$, then $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is linearly dependent if and only if $\text{det}(A) = 0$.

**Example 12.** Determine whether the given set of vectors is linearly independent or dependent in $\mathbb{R}^n$. In the case of linear dependence, find the dependency relationship.
(a) $\{(1, 2, 1), (1, 0, -1), (1, 1, 1)\}$  
(b) $\{(1, 1, 1), (1, -1, 2), (3, 1, 4)\}$
(c) $\{(1, 2), (1, -3), (9, 0)\}$
Example 13. Determine whether the given set of vectors is linearly independent in $M_2(\mathbb{R})$.

\[ A_1 = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}, \quad \text{and} \quad A_3 = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}. \]

**Definition.** The set of functions $\{f_1, f_2, \dots, f_k\}$ is linearly independent on an interval $I$ if and only if the only values of the scalars $a_1, a_2, \dots, a_k \in \mathbb{F}$ such that

\[ a_1 f_1(x) + a_2 f_2(x) + \cdots + a_k f_k(x) = 0, \quad \forall x \in I, \]

are $a_1 = a_2 = \cdots = a_k = 0$.

**Definition.** Let $f_1, f_2, \dots, f_k$ be functions in $C^{k-1}(I)$. The Wronskian of these functions is the order $k$ determinant defined by

\[
W[f_1, f_2, \ldots, f_k](x) = \begin{vmatrix}
 f_1(x) & f_2(x) & \cdots & f_k(x) \\
 f'_1(x) & f'_2(x) & \cdots & f'_k(x) \\
 \vdots & \vdots & \ddots & \vdots \\
 f^{(k-1)}_1(x) & f^{(k-1)}_2(x) & \cdots & f^{(k-1)}_k(x)
\end{vmatrix}.
\]

**Theorem 4.5.5.** Let $f_1, f_2, \ldots, f_k$ be functions in $C^{k-1}(I)$. If $W[f_1, f_2, \ldots, f_k]$ is nonzero at some point $x_0 \in I$, then $\{f_1, f_2, \ldots, f_k\}$ is linearly independent of $I$.

**Note.** If $W[f_1, f_2, \ldots, f_k](x) = 0 \ \forall x \in I$, then the previous theorem gives no information as to linear dependence or independence of $\{f_1, f_2, \ldots, f_k\}$ on $I$. 

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Example 14. Use the Wronskian to show that the given functions are linearly independent on the given interval.

\[ f_1(x) = \sin x, \quad f_2(x) = \cos x, \quad f_3(x) = \tan x, \quad \text{and} \quad I = \left( -\frac{\pi}{2}, \frac{\pi}{2} \right). \]
4.6 Bases and Dimension

**Definition.** A set of vectors \( \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \} \) in a vector space \( V \) is called a **basis** for \( V \) if

1. The vectors are linearly independent.
2. The vectors span \( V \).

**Example 15.** Show that \( \{(1,0), (0,1)\} \) is a basis for \( \mathbb{R}^2 \). We call this basis the **standard basis**.

---

**Note. Notation**

- In \( \mathbb{R}^n \), let \( \vec{e}_i \) be the vector with value 1 in the \( i \)th position and zeros elsewhere.
  
  So for \( \mathbb{R}^2 \), \( \vec{e}_1 = (1,0) \) and \( \vec{e}_2 = (0,1) \).

- Let \( E_{ij} \) denote the \( m \times n \) matrix with the value 1 in the \((i,j)\)-position and zeroes elsewhere.

---

**Theorem 4.6.1.** If a finite-dimensional vector space has a basis consisting of \( m \) vectors, then any set of more than \( m \) vectors is linearly dependent.

**Corollary 4.6.1.** All bases in a finite-dimensional vectors space \( V \) contain the same number of vectors.

**Proof:**
Corollary 4.6.2. If a finite-dimensional vector space $V$ has a basis consisting of $n$ vectors, then any spanning set must contain at least $n$ vectors.

Example 16. Determine whether the given set of vectors is a basis of $\mathbb{R}^3$.
$\{(2, 2, 2), (0, 0, 3), (0, 1, 1)\}$

Definition. The dimension of a finite-dimensional vector space $V$, written $\dim[V]$, is the number of vectors in any basis for $V$. If $V$ is the trivial vector space, $V = \{\vec{0}\}$, then we define its dimension to be zero.

Note. The following dimensions should be remembered:

$\dim[\mathbb{R}^n] = n$, $\dim[M_{m \times n}(\mathbb{R})] = mn$, $\dim[M_n(\mathbb{R})] = n^2$, $\dim[P_n] = n + 1$
**Theorem 4.6.2.** If $\dim[V] = n$, then any set of $n$ linearly independent vectors in $V$ is a basis for $V$.

**Example 17.** State two possible basis for $\mathbb{R}^2$.

**Theorem 4.6.3.** If $\dim[V] = n$, then any set of $n$ vectors in $V$ that span $V$ is a basis for $V$.

**Corollary 4.6.3.** If $\dim[V] = n$ and $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a set of $n$ vectors in $V$, the following statements are equivalent:

1. $S$ is a basis for $V$.
2. $S$ is linearly independent.
Corollary 4.6.4. Let $S$ be a subspace of a finite-dimensional vector space $V$. If $\dim[V] = n$, then $\dim[S] \leq n$.

Furthermore, if $\dim[S] = n$, then $S = V$.

Theorem 4.6.4. Let $S$ be a subspace of a finite-dimensional vector space $V$. Any basis for $S$ is part of a basis for $V$.

Proof:

Example 18. Let $S$ be the subspace of $\mathbb{R}^3$ consisting of all vectors of the form $(3r - 2s, r, 4r + 5s)$ where $r, s \in \mathbb{R}$. Determine a basis for $S$ and hence, find $\dim[S]$. 

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Example 19. Let $S$ denote the subspace of $M_2(\mathbb{R})$ consisting of all symmetric $2 \times 2$ matrices. Determine a basis for $S$, and find $\text{dim}[S]$. 
5 Linear Transformation

5.6 The Eigenvalue/Eigenvector Problem

**Definition.** Let $A$ be an $n \times n$ matrix. Any values of $\lambda$ for which $A\vec{v} = \lambda \vec{v}$ has nontrivial solutions $\vec{v}$ are called eigenvalues of $A$. The corresponding nonzero vectors $\vec{v}$ are called eigenvectors of $A$.

**Note.**
- Matrix $A$ will be interpreted as a linear transformation $T : \mathbb{C}^n \rightarrow \mathbb{C}^n$ in the usual manner, that is, $T(\vec{v}) = A\vec{v}$.
- Eigenvalues are sometimes referred to as characteristic values while their respective eigenvectors are referred to as characteristic vectors of $A$.
- Eigenvalues and eigenvectors come in pairs. Each eigenvector has an associated eigenvalue.

Let’s look at a geometric interpretation:

**Example 1.** Verity that $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and find its associated eigenvalue.
Steps for finding Eigenvalues and Eigenvectors of a given $n \times n$ matrix $A$:

1. Find all scalars $\lambda$ such that $\det(A - \lambda I) = 0$. These are the eigenvalues of $A$.

2. For each eigenvalue $\lambda$, solve the equation $(A - \lambda I)\vec{v} = \vec{0}$ to find its associated eigenvector.

**Definition.** For a given $n \times n$ matrix $A$, the polynomial $p(\lambda)$ defined by

$$p(\lambda) = \det(A - \lambda I)$$

is called the **characteristic polynomial** of $A$, and the equation

$$p(\lambda) = 0$$

is called the **characteristic equation** of $A$.

**Example 2.** Find the characteristic polynomial, characteristic equation, and finally eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.
Theorem 5.6.1. Let $A$ be an $n \times n$ matrix with real elements. If $\lambda$ is a complex eigenvalue of $A$ with corresponding eigenvector $\vec{v}$, then $\bar{\lambda}$ is an eigenvalue of $A$ with corresponding eigenvector $\bar{\vec{v}}$.

Proof:

Example 3. Find all the eigenvalues and eigenvectors of $A = \begin{bmatrix} 6 & -13 \\ 1 & 0 \end{bmatrix}$. 
6 Linear Differential Equations of Order \( n \)

6.1 General Theory for Linear Differential Equations

**Definition.** The general \( n \)th-order linear differential equation is

\[
a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x)
\]

where \( a_0, a_1, a_2, \ldots, a_n, F, a_0 \neq 0 \), are functions defined on an interval \( I \).

\( D \) is called a **derivative operator** for a differential function \( f \), \( D^k(f) = \frac{d^k f}{dx^k} \).

By taking a linear combination of the basic derivative operators, we obtain the general **linear differential operator of order** \( n \),

\[
L = D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n,
\]

defined by

\[
Ly = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y,
\]

where \( a_i \) is a function of \( x \) for all \( i = 1, 2, \ldots, n \).

**Note.** For \( y_1, y_2 \in C^n(I) \) and all scalars,

\[
L(y_1 + y_2) = Ly_1 + Ly_2,
\]

\[
L(c y) = cL(y).
\]

*Recall: \( C^k(I) \) is the vector space of all real-valued functions that are continuous and have (at least) \( k \) continuous derivatives on \( I \).*

**Example 1.** Find \( Ly \) for \( L = D^2 - x^2 D + x \) if \( y(x) = 4 \tan^2 x \).
Definition. If all functions $a_i$ for $i = 1, 2, \ldots, n$ and $F$ are continuous on the interval $I$, the equation is said to be regular.

*We will only be learning about regular differential equations in this course.*

Definition. $a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x)$, where $a_0 \neq 0$, is homogeneous provided $F(x) = 0$ on an interval $I$. Otherwise, it is called nonhomogeneous.

Note. Let $L$ be a linear differential operator of order $n$. Then since $a_0 \neq 0$, then we can divide by it and therefore the general form of a differential equation can be written as $Ly = F(x)$.

**Theorem 6.1.1.** Let $a_1, a_2, \ldots, a_n$ and $F$ be functions that are continuous on an interval $I$. Then, for any $x_0 \in I$, the initial-value problem

$$Ly = F(x),$$

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}$$

has a unique solution $I$.

Definition. Let $T : V \rightarrow W$ be a linear transformation. Then the kernel of $T$, denoted by $Ker(T)$, is defined as $Ker(T) = \{ \vec{v} \in V | T(\vec{v}) = \vec{0} \}$.

Note. This means that the solution set of a homogeneous differential equation is precisely the kernel of $L$ such that $Ly = 0$ where $L : C^n(I) \rightarrow C^0(I)$.

Definition. $S = \{ y \in C^n(I) | Ly = 0 \}$ or $S = Ker(L)$ is the subspace of $C^n(I)$ that is the set of all solutions to a particular homogeneous linear differential solution, refer to as the solution space.

**Example 2.** Verify that $y(x) = xe^x$ is in the kernel of $L$ where $L = -D^2 + 2D - 1$. 

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Example 3. Compute $\text{Ker}(L)$ for $L = x^2D + x$.

**Theorem 6.1.2.** The set of all solutions to the regular $n$th-order homogeneous linear differential equation

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0$$

on an interval $I$ is a vector space of dimension $n$.

**Note. Implication of the Previous Theorem:** There are $n$ linearly independent solutions to homogeneous linear differential equations of order $n$.

**Theorem 6.1.3.** Let $y_1, y_2, \ldots, y_n$ be solutions to the regular $n$th-order homogeneous linear differential equation $Ly = 0$ on an interval $I$, and let $W[y_1, y_2, \ldots, y_n](x)$ denote their Wronskian. If $W[y_1, y_2, \ldots, y_n](x_0) = 0$ for any $x_0 \in I$, then $\{y_1, y_2, \ldots, y_n\}$ is linearly dependent on $I$.

Recall:

**Definition.** Let $f_1, f_2, \ldots, f_k$ be functions in $C^{k-1}(I)$. The **Wronskian** of these functions is the order $k$ determinant defined by

$$W[f_1, f_2, \ldots, f_k](x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_k(x) \\ f_1'(x) & f_2'(x) & \cdots & f_k'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(k-1)}(x) & f_2^{(k-1)}(x) & \cdots & f_k^{(k-1)}(x) \end{vmatrix}.$$ 

**Example 4.** (a) Determine three solutions to $y''' + y'' - 10y' + 8y = 0$ of the form $y(x) = e^{rx}$.
(b) Show that the resulting solutions from (a) are linearly independent and thereby determine the general solution to the differential equation.

**Definition.** For a given nonhomogeneous linear differential equation

\[ y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x), \quad F(x) \neq 0, \]

if we set \( F(x) = 0 \), the result is the **associated homogeneous equation**.

**Theorem 6.1.4.** Let \( \{y_1, y_2, \ldots, y_n\} \) be a linearly independent set of solutions to \( Ly = 0 \) on an interval \( I \), and let \( y = y_p \) be any particular solution to \( Ly = F \) on \( I \). Then every solution to \( Ly = F \) on \( I \) is of the form

\[ y = c_1y_1 + c_2y_2 + \cdots + c_ny_n + y_p \]

for appropriate constants \( c_1, c_2, \ldots, c_n \).

**Proof:**

\[ \]
**Example 5.** Determine a particular solution to \( y'''' + y'' - 10y' + 8y = e^{-2x} \) of the form \( y_p(x) = A_0 e^{-2x} \). Then find the general solution to the differential equation.

**Theorem 6.1.5.** If \( y = u_p \) and \( y = v_p \) are particular solutions of \( Ly = f(x) \) and \( Ly = g(x) \), respectively, then \( y = u_p + v_p \) is a solution to \( Ly = f(x) + g(x) \).
6.2 Constant-Coefficient Homogeneous Linear Differential Equations

**Definition.** A constant-coefficient homogeneous linear differential equation is a homogeneous linear differential equation of the form

\[ y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0, \]

where \( a_1, a_2, \ldots, a_n \) are constants.

**Definition.** \( P(D) \) where \( P(D) = D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n \), where \( P(D) \) is called the polynomial differential operator.

Associated with any polynomial differential operator is the real polynomial

\[ P(r) = r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n, \]

referred to as the auxiliary polynomial.

The corresponding polynomial equation \( P(r) = 0 \) is called the auxiliary equation.

**Theorem 6.2.1.** If \( P(D) \) and \( Q(D) \) are polynomial differential operators, then

\[ P(D)Q(D) = Q(D)P(D). \]

**Note.** In general, linear transformations are NOT commutative, so this is a significant result.

**Theorem 6.2.2.** If \( P(D) = P_1(D)P_2(D) \cdots P_k(D) \), where each \( P_i(D) \) is a polynomial differential operator, then, for \( 1 \leq i \leq k \), any solution to \( P_i(D)y = 0 \) is also a solution to \( P(D)y = 0 \).

**Note.** Recall:

(a) Euler’s Formula:

\[ e^{(a+ib)x} = e^{ax}[\cos(bx) + i \sin(bx)]. \]

(b) If \( r = a + ib \), then

\[ \frac{d}{dx}(e^{rx}) = re^{rx}. \]
**Theorem 6.2.3.** The differential equation \((D - r)^m y = 0\), where \(m\) is a positive integer and \(r\) is real or complex number, has the following \(m\) solutions that are linearly independent on any interval:

\[
e^{rx}, x e^{rx}, x^2 e^{rx}, \ldots, x^{m-1} e^{rx}.
\]

**Theorem 6.2.4.** Consider the differential equation \(P(D)y = 0\). Let \(r_1, r_2, \ldots, r_k\) be distinct roots of the auxiliary equation so that

\[
P(r) = (r - r_1)^{m_1}(r - r_2)^{m_2}\cdots(r - r_k)^{m_k},
\]

where \(m_i\) denotes the multiplicity of the root \(r = r_i\).

1. If \(r_i \in \mathbb{R}\), then the function \(e^{r_1 x}, x e^{r_1 x}, \ldots, x^{m_i-1} e^{r_1 x}\) are linearly independent solutions to \(P(D)y = 0\) on any interval.

2. If \(r_j\) is complex, say \(r_j = a + ib\) for \(a, b \in \mathbb{R}\) and \(b \neq 0\), then the functions

\[
e^{ax} \cos(bx), x e^{ax} \cos(bx), \ldots, x^{m_j-1} e^{ax} \cos(bx),
\]

\[
e^{ax} \sin(bx), x e^{ax} \sin(bx), \ldots, x^{m_j-1} e^{ax} \sin(bx)
\]

corresponding to the conjugate roots \(r = a \pm ib\) are linearly independent solutions to \(P(D)y = 0\) on any interval.

3. The \(n\) real-valued solutions \(y_1, y_2, \ldots, y_n\) to \(P(D)y = 0\) that are obtained by considering the distinct roots \(r_1, r_2, \ldots, r_k\) are linearly independent on any interval. So, the general solution to \(P(D)y = 0\) is

\[
y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x).
\]

**Example 6.** Find the general solution to the given differential equation.

(a) \((D^2 - 2)y = 0\)

(b) \((D^2 + 2D + 10)^2 y = 0\).
Example 7. Solve \( y''' - y'' + y' - y = 0 \) given \( y(0) = 0, \ y'(0) = 1, \ y''(0) = 2. \)
6.3 The Method of Undetermined Coefficients: Annihilators

Recall: The general solution to the nonhomogeneous differential equation $P(D)y = F(x)$ is of the form $y(x) = y_c(x) + y_p(x)$. A solution to $y_p(x)$ is called a trial solution.

**Note.** Now we move on to finding solutions to nonhomogeneous differential equations.

**Method of Undetermined Coefficients**

- The differential equation has constant coefficients and is therefore of the form $P(D)y = F(x)$.
- There exist a polynomial differential operator $A(D)$ such that $A(D)F = 0$.

**Definition.** Any polynomial differential operator $A(D)$ that satisfies $A(D)F = 0$ is said to annihilate $F(x)$.

The polynomial differential operator of lowest order that satisfies $A(D)F = 0$ is called the annihilator of $F$.

**Example 8.** Determine the annihilator of $F(x) = 5e^{-3x}$.

\[
A(D) = (D - a)^{k+1} \text{ annihilates each of the functions } e^{ax}, xe^{ax}, \ldots, x^ke^{ax},
\]

and so it also annihilates

\[
F(x) = (a_0 + a_1x + \cdots + a_kx^k)e^{ax}.
\]
Example 9. Determine the annihilator for $F(x) = 4e^{-2x} \sin x$.

$$A(D) = D^2 - 2aD + a^2 + b^2$$ annihilates both of the functions

$$e^{ax} \cos(bx) \text{ and } e^{ax} \sin(bx),$$

and therefore it also annihilates

$$F(x) = e^{ax}[a_0 \cos(bx) + b_0 \sin(bx)]$$

for all values of the constants $a_0$, $b_0$. So

$$A(D) = D^2 + b^2$$

in the case where $a = 0$ or equivalently, the functions $\cos(bx)$ and $\sin(bx)$. 
Example 10. Determine the annihilator for $F(x) = (1 - 3x)e^{4x} + 2x^2$.

$A(D) = D^{k+1}$ annihilates $F(x) = a_0 + a_1x + \cdots + a_kx^k$.

Example 11. Determine the annihilator for $F(x) = x^2 \sin x$. 


\[ A(D) = (D^2 - 2aD + a^2 + b^2)^{k+1} \] annihilates the functions

\[ e^{ax} \cos(bx), \ xe^{ax} \cos(bx), \ x^2 e^{ax} \cos(bx), \ldots, \ x^k e^{ax} \cos(bx), \]

\[ e^{ax} \sin(bx), \ xe^{ax} \sin(bx), \ x^2 e^{ax} \sin(bx), \ldots, \ x^k e^{ax} \sin(bx), \]

and for all values of constants \( a_0, a_1, \ldots, a_k, b_0, b_1, \ldots, b_k \), it annihilates

\[ F(x) = (a_0 + a_1 x + \cdots + a_k x^k)e^{ax} \cos(bx) + (b_0 + b_1 x + \cdots + b_k x^k)e^{ax} \sin(bx). \]

If \( F(x) \) is a sum of the functions given in the previous statements, then \( F(x) \) is annihilated by the corresponding PRODUCT of the annihilators in the previous statements.

**Note.** The choice of trial solutions will depend on the specific case. For a list of these cases and appropriate trial solutions look at pages 479 - 480 in your book.

**Example 12.** Determine the general solution to the given differential equation.

(a) \( y''' + 3y'' + 3y' + y = 2e^{-x} + 3e^{2x} \)
(b) Determine an appropriate trial solution for \((D^2 - 2D + 2)^3(D - 2)^2(D + 4)y = e^x \cos x - 3e^{2x}\).
6.6 RLC Circuits

Recall:

<table>
<thead>
<tr>
<th>Kirchhoff’s Second Law</th>
</tr>
</thead>
<tbody>
<tr>
<td>The sum of the voltage drops around a closed circuit is zero. In other words, around any loop in a circuit, whatever energy a charge starts with, it must end up losing all of that energy by the time it gets to the end. Mathematically, this is equivalent to</td>
</tr>
<tr>
<td>[ \Delta V_R + \Delta V_C + \Delta V_L - E(t) = 0. ]</td>
</tr>
</tbody>
</table>

**Note.** We already done problems involving RL and RC circuits, and now we focus on LC and RLC circuits.

Recall:

\[ \Delta V_R + \Delta V_C + \Delta V_L - E(t) = 0 \]

becomes

\[ \frac{di}{dt} + \frac{R}{L}i + \frac{1}{LC}q = \frac{1}{L}E(t). \]

This leads to

\[ \frac{d^2q}{dt^2} + \frac{R}{L} \frac{dq}{dt} + \frac{1}{LC}q = \frac{1}{L}E(t), \]

which has an associated auxiliary equation

\[ r^2 + \frac{R}{L}r + \frac{1}{LC} = 0. \]

Which then can be solve by using the techniques we just learned.

**Circuit cases**

1. **LC Circuit:** This is the case when there is no resistance.

**Example 13.** What is the resulting equation for the LC circuit case?
Example 14. Find the charge for $t > 0$ in the $LC$ circuit with an inductance of $2 \, H$, a capacitor of $\frac{1}{32} \, F$, an $E(t) = 6t^2$, and initial condition $q(0) = 3$ and $q'(0) = 7$. 
2. *RLC* Circuit: The circuit includes resistance, inductance, and capacitance.

**Example 15.** Find the charge for $t > 0$ in the *RLC* circuit with an inductance of $\frac{1}{4} \text{H}$, a resistance of $4 \text{Ω}$, a capacitance of $\frac{1}{16} \text{F}$, an $E(t) = 6e^t$, and initial condition $q(0) = 2$ and $q'(0) = -14$. 


6.7 The Variation-of-Parameters Method

Limitations of the Method of Undetermined Coefficients

- It can only be used for differential equations with constant coefficients.
- The nonhomogeneous terms are limited to linear combinations of $cx^ke^{ax}$, $cx^ke^{ax}\cos(bx)$, $cx^ke^{ax}\sin(bx)$ and $cx^k$.

Theorem 6.7.1. Variation-of-Parameters Theorem:

Consider $y'' + a_1y' + a_2y = F$, where $a_1$, $a_2$ and $F$ are at least continuous on an interval $I$. Let $y_1$ and $y_2$ be linearly independent solutions to the associated homogeneous equation

$$y'' + a_1y' + a_2y = 0$$

on $I$. Then the particular solution to the equation

$$y'' + a_1y' + a_2y = F$$

is $y_p = u_1y_1 + u_2y_2$ where $u_1$ and $u_2$ satisfy

$$y_1u_1' + y_2u_2' = 0$$

and

$$y_1u_1'' + y_2u_2'' = F.$$
Example 16. Find the general solution to \( y'' - 2my' + m^2y = \frac{e^{mx}}{1 + x^2} \) where \( m \) is a constant.
Example 17. Find the general solution to \( y'' + y = \csc x + 2x^2 + 5x + 1 \) where \( 0 < x < \pi \).
Note. In the previous section, the coefficients were still constant. Here we deal with nonconstant coefficients.

**Definition.** A differential equation of the form
\[ x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = 0 \]
where \( a_1, a_2, \ldots, a_n \) are constants is called a **Cauchy-Euler equation**.

Suppose we have the Cauchy-Euler equation \( x^2 \frac{d^2 y}{dx^2} + a_1 x \frac{dy}{dx} + a_2 y = 0 \), let \( y = x^r \). So...
### Roots of Indicial equation

<table>
<thead>
<tr>
<th>Roots of Indicial equation</th>
<th>Linearly Independent Solutions to a Differential Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>real distinct roots:</td>
<td>( y_1(x) = x^{r_1} ) and ( y_2(x) = x^{r_2} )</td>
</tr>
<tr>
<td>( r_1 \neq r_2 )</td>
<td></td>
</tr>
<tr>
<td>real repeated roots:</td>
<td>( y_1(x) = x^{r_1} ) and ( y_2(x) = x^{r_2} \ln x )</td>
</tr>
<tr>
<td>( r_1 = r_2 )</td>
<td></td>
</tr>
<tr>
<td>complex conjugate roots:</td>
<td>( y_1(x) = x^a \cos(b \ln x) ) and ( y_2(x) = x^a \sin(b \ln x) )</td>
</tr>
<tr>
<td>( r_1 = a + ib ) and ( r_2 = a - ib )</td>
<td></td>
</tr>
</tbody>
</table>

**Example 18.** Find the general solution to \( x^2y'' - x(2m - 1)y' + m^2y = 0 \) where \( m \) is a constant and \( x \in (0, \infty) \).  

**Example 19.** Find the general solution to \( x^2y'' - xy' + 5y = 0 \) on the interval \((0, \infty)\).
Example 20. Find the general solution to \( x^2y'' - xy' + 5y = 8x(\ln x)^2 \) on the interval \((0, \infty)\).
6.9 Reduction of Order

**Theorem 6.9.1. Reduction of Order Theorem:**

If \( y = y_1(x) \) is a solution to

\[
y'' + a_1(x)y' + a_2(x)y = 0
\]

on an interval \( I \), then substituting \( y(x) = y_1(x)u(x) \) into

\[
y'' + a_1(x)y' + a_2(x)y = F(x)
\]

yields its general solution.
Example 21. Find a second linearly independent solution and the general solution to

\[ y'' - x^{-1}y' + 4x^2y = 0, \quad x > 0, \]

given one solutions \( y_1(x) = \sin(x^2) \).
Example 22. Find the general solution to $xy'' - (2x + 1)y' + 2y = 8x^2e^{2x}$, for $x > 0$, given one solution $y_1(x) = e^{2x}$ to the associated homogeneous equation.
7 System of Differential Equations

7.1 First-Order Linear System

**Definition.** A system of differential equations of the form

\[
\frac{dx_1}{dt} = a_{11}(t)x_1(t) + \cdots + a_{1n}(t)x_n(t) + b_1(t), \\
\frac{dx_2}{dt} = a_{21}(t)x_1(t) + \cdots + a_{2n}(t)x_n(t) + b_2(t), \\
\vdots \\
\frac{dx_n}{dt} = a_{n1}(t)x_1(t) + \cdots + a_{nn}(t)x_n(t) + b_n(t),
\]

where the \( a_{ij}(t) \) and \( b_i(t) \) are specified functions on an interval \( I \), is called a first-order linear system. If \( b_1 = b_2 = \cdots = b_n = 0 \), then the system is called homogeneous. Otherwise, it is called nonhomogeneous. Note that \( \frac{dx_i}{dt} \) will be denoted as \( x_i' \).

**Definition.** A solution to the system above on an interval \( I \) is an ordered \( n \)-tuple of functions \( x_1(t), x_2(t), \ldots, x_n(t) \), which, when substituted into both sides of the system, yield the same result \( \forall t \in I \).

**Definition.** Solving a first order-linear system subject to \( n \) auxiliary conditions imposed at the same value of the independent variable is called an initial-value problem. Thus, the general form of the auxiliary conditions for an initial-value problem is:

\[
x_1(t_0) = \alpha_1, \; x_2(t_0) = \alpha_2, \; \ldots, \; x_n(t_0) = \alpha_n,
\]

where \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are constants.

**Example 1.** Solve the given system of differential equations. \( x_1' = 2x_1, \; x_2' = x_2 - x_3, \; x_3' = x_2 + x_3 \)
Example 2. Solve the given initial-value problem.

\[ x'_1 = 2x_1 + x_2, \quad x'_2 = -x_1 + 4x_2, \quad x_1(0) = 1, \quad x_2(0) = 3 \]
Example 3. Convert the following system of differential equations to a first-order linear system.

$$\frac{dx}{dt} - ty = \cos t, \quad \frac{d^2y}{dt^2} - \frac{dx}{dt} + x = e^t$$

Example 4. Convert the given linear differential equation to a first-order linear system.

$$y''' + t^2 y' - e^t y = t$$
### 7.2 Vector Formulation

\[
\begin{align*}
    x_1' &= a_{11}(t)x_1(t) + \cdots + a_{1n}(t)x_n(t) + b_1(t), \\
    x_2' &= a_{21}(t)x_1(t) + \cdots + a_{2n}(t)x_n(t) + b_2(t), \\
    &\vdots \\
    x_n' &= a_{n1}(t)x_1(t) + \cdots + a_{nn}(t)x_n(t) + b_n(t),
\end{align*}
\]

Can be written as the equivalent vector equation, also known as a vector differential equation,

\[
\vec{x}'(t) = A(t)\vec{x}(t) + \vec{b}(t),
\]

where,

\[
\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad \vec{x}'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}, \quad A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}, \quad \text{and}
\]

\[
\vec{b}(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{bmatrix}.
\]

**Note.** Let \( V_n(I) \) denote the set of all column \( n \)-vector functions defined on an interval \( I \), and define addition and scalar multiplication within this set in the same manner as for column vectors.

**Theorem 7.2.1.** The set \( V_n(I) \) is a vector space.

**Definition.** Let \( \vec{x}_1(t), \vec{x}_2(t), \ldots, \vec{x}_n(t) \) be vectors in \( V_n(I) \). Then the Wronskian of these vector functions, denoted \( W[\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n](t) \), is defined by

\[
W[\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n](t) = \det([\vec{x}_1(t), \vec{x}_2(t), \ldots, \vec{x}_n(t)]).
\]

**Example 5.** Determine the Wronskian of the given vector functions.

\[
\vec{x}_1(t) = \begin{bmatrix} t \\ t \end{bmatrix}, \quad \vec{x}_2(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix}
\]
Theorem 7.2.2. Let $\vec{x}_1(t), \vec{x}_2(t), \ldots, \vec{x}_n(t)$ be vectors in $V_n(I)$. If $W[\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n](t_0)$ is nonzero at some point $t_0 \in I$, then $\{\vec{x}_1(t), \vec{x}_2(t), \ldots, \vec{x}_n(t)\}$ is linearly independent on $I$.

Example 6. Determine if the given vector functions are linearly independent on $(-\infty, \infty)$.

(a) $\vec{x}_1(t) = \begin{bmatrix} t \\ t \end{bmatrix}$, $\vec{x}_2(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix}$.

(b) $\vec{x}_1(t) = \begin{bmatrix} t^2 \\ 6 - t + t^2 \end{bmatrix}$, $\vec{x}_2(t) = \begin{bmatrix} -3t^2 \\ -18 + 3t - 3t^2 \end{bmatrix}$.
7.3 General Results for First-Order Linear Differential Systems

**Theorem 7.3.1.** The initial value-problem

\[ \ddot{x}'(t) = A(t) \ddot{x}(t) + \ddot{b}(t), \quad \ddot{x}(t_0) = \ddot{x}_0, \]

where \( A(t) \) and \( \ddot{b}(t) \) are continuous on an interval \( I \), has a unique solution on \( I \).

**Theorem 7.3.2.** The set of all solutions \( \ddot{x}'(t) = A(t) \ddot{x}(t) \), where \( A(t) \) is an \( n \times n \) matrix function that is continuous on an interval \( I \), is a vector space of dimension \( n \).

**Note.** It follows from the theorem above that any set of \( n \) linearly independent solutions can be written as

\[ \ddot{x}(t) = c_1 \ddot{x}_1(t) + c_2 \ddot{x}_2(t) + \cdots + c_n \ddot{x}_n(t), \]

for appropriate constants \( c_1, c_2, \ldots, c_n \).

**Definition.** Let \( A(t) \) be an \( n \times n \) matrix function that is continuous on an interval \( I \). Any set of \( n \) solutions, \( \{\ddot{x}_1, \ddot{x}_2, \ldots, \ddot{x}_n\} \), to \( \ddot{x}' = A\ddot{x} \) that is linearly independent on \( I \) is called a **fundamental solution set** on \( I \). The corresponding matrix \( X(t) \) defined by

\[ X(t) = [\ddot{x}_1, \ddot{x}_2, \ldots, \ddot{x}_n] \]

is called a **fundamental matrix** for the vector differential equation \( \ddot{x}' = A\ddot{x} \).

**Theorem 7.3.3.** Let \( A(t) \) be an \( n \times n \) matrix function that is continuous on an interval \( I \), If \( \{\ddot{x}_1, \ddot{x}_2, \ldots, \ddot{x}_n\} \) is a linearly independent set of solutions to \( \ddot{x}' = A\ddot{x} \) on \( I \), then

\[ W[\ddot{x}_1, \ddot{x}_2, \ldots, \ddot{x}_n](t) \neq 0 \]

at every point \( t \in I \).

**Proof:**

\[ \]
Example 7. Show that the given functions are solutions of the system \( \ddot{x}'(t) = A(t) \ddot{x}(t) \) for the given matrix \( A \), and hence, find the general solution to the system.

\[
\ddot{x}_1(t) = \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix}, \quad \ddot{x}_2(t) = \begin{bmatrix} e^{2t}(1 + t) \\ -te^{2t} \end{bmatrix}, \quad A = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}.
\]

Theorem 7.3.4. Let \( A(t) \) be a matrix function that is continuous on an interval \( I \), and let \( \{ \ddot{x}_1, \ddot{x}_2, \ldots, \ddot{x}_n \} \) be a fundamental solution set on \( I \) for the vector differential equation \( \ddot{x}'(t) = A(t) \ddot{x}(t) \). If \( \ddot{x} = \ddot{x}_p(t) \) is any particular solution to the nonhomogeneous vector differential equation

\[
\ddot{x}'(t) = A(t) \ddot{x}(t) + \ddot{b}(t)
\]
on \( I \), then every solution to the equation above on \( I \) is of the form

\[
\ddot{x}(t) = c_1 \ddot{x}_1 + c_2 \ddot{x}_2 + \cdots + c_n \ddot{x}_n + \ddot{x}_p.
\]
7.4 Vector Differential Equations: Nondefective Coefficient Matrix

**Theorem 7.4.1.** Let \( A \) be an \( n \times n \) matrix of real constants, and let \( \lambda \) be an eigenvalue of \( A \) with corresponding eigenvector \( \vec{v} \). Then,

\[
\vec{x}(t) = e^{\lambda t} \vec{v}
\]

is a solution to the constant-coefficient vector differential equation \( \vec{x}' = A \vec{x} \) on any interval.

**Note.** This result also holds for complex eigenvalues and eigenvectors.

**Definition.** An \( n \times n \) matrix \( A \) that has \( n \) linearly independent eigenvectors is called nondefective. In such case, we say that \( A \) has a complete set of eigenvectors. If \( A \) has less than \( n \) linearly independent eigenvectors, it is called defective.

**Theorem 7.4.2.** Let \( A \) be an \( n \times n \) matrix of real constants. If \( A \) has \( n \) real linearly independent eigenvectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \), with corresponding real eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \) (not necessarily distinct), then the vector functions \( \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\} \) defined by

\[
\vec{x}_k(t) = e^{\lambda_k t} \vec{v}_k, \quad k = 1, 2, \ldots, n,
\]

\( \forall t \), are linearly independent solutions to \( \vec{x}' = A \vec{x} \) on any interval. The general solution to this vector differential equation is

\[
\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 + \cdots + c_n \vec{x}_n.
\]

**Proof**
Example 8. Find the general solution to

\begin{align*}
x_1' &= -x_1 \\
x_2' &= x_1 + 5x_2 - x_3 \\
x_3' &= x_1 + 6x_2 - 2x_3
\end{align*}
Theorem 7.4.3. Let $\mathbf{u}(t)$ and $\mathbf{v}(t)$ be real-valued vector functions. If

$$\mathbf{w}_1(t) = \mathbf{u}(t) + i \mathbf{v}(t) \quad \text{and} \quad \mathbf{w}_2(t) = \mathbf{u}(t) - i \mathbf{v}(t)$$

are complex conjugate solutions to $\mathbf{x}' = A \mathbf{x}$, then

$$\mathbf{x}_1(t) = \mathbf{u}(t) \quad \text{and} \quad \mathbf{x}_2(t) = \mathbf{v}(t)$$

are themselves real-valued solutions of $\mathbf{x}' = A \mathbf{x}$.

Example 9. Determine the general solution to the system $\mathbf{x}' = A \mathbf{x}$ for the given matrix $A$.

$$A = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}.$$
8 The Laplace Transform and Some Elementary Applications

8.1 Definition of the Laplace Transform

**Definition.** Let \( f \) be a function defined on the interval \([0, \infty)\). The **Laplace transform** of \( f \) is the function \( F(s) \) defined by

\[
F(s) = \int_0^\infty e^{-st} f(t) \, dt,
\]

provided that the improper integral converges. We will usually denote the Laplace transform of \( f \) by \( \mathcal{L}[f] \).

**Note.** The Laplace transform is a technique for transforming ordinary differential equations into algebraic equations. It is a tool that will simplify the calculations used in circuits, spring problems, and other various mechanical systems.

**Example 1.** Determine the \( \mathcal{L}[f] \) for \( f(t) = 3e^{2t} \).
Suppose that the Laplace transforms of both $f$ and $g$ exist for $s > \alpha$, where $\alpha \in \mathbb{R}$. Furthermore, let $c \in \mathbb{R}$. Then

1. $L[f + g] =$
2. $L[cf] =$

**Example 2.** Determine $L[f]$ for $f(t) = 2 \sin^2(4t) - 3$. 

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**Common Laplace Transforms:** Let \( n \in \mathbb{N}, a, b \in \mathbb{R} \).

- \( L[t^n] = \frac{n!}{s^{n+1}}, \quad s > 0 \)
- \( L[e^{at}] = \frac{1}{s - a}, \quad s > a \)
- \( L[\cos(bt)] = \frac{s}{s^2 + b^2}, \quad s > 0 \)
- \( L[\sin(bt)] = \frac{b}{s^2 + b^2}, \quad s > 0 \)

**Definition.** A function \( f \) is called **piecewise continuous** on the interval \([a, b]\) if we can divide \([a, b]\) into a finite number of subintervals in such a manner that

1. \( f \) is continuous on each subinterval, and
2. \( f \) approaches a finite limit as the endpoints of each subinterval are approached from within.

If \( f \) is piecewise continuous on every interval of the form \([0, b]\), where \( b \) is a constant, then we say that \( f \) is piecewise continuous on \([0, \infty)\).

**Example 3.** Sketch the given functions and determine its Laplace transform \( f(t) = \begin{cases} 1, & 0 \leq t \leq 2 \\ -1, & t > 2 \end{cases} \).
8.2 The Existence of the Laplace Transform and the Inverse Transform

**Definition.** A function \( f \) is said to be of **exponential order** if there exist constants \( M \) and \( \alpha \) such that
\[
|f(t)| \leq Me^{\alpha t}, \quad \forall t > 0.
\]

**Example 4.** Show that \( f(t) = 4 \sin(4t) \) is of exponential order.

**Lemma 8.2.1.** The Comparison Test for Improper Integrals:
Suppose that \( 0 \leq G(t) \leq H(t) \) for \( 0 \leq t < \infty \). If \( \int_0^\infty H(t) \, dt \) converges, then \( \int_0^\infty G(t) \, dt \) converges.

**Theorem 8.2.1. Laplace Transform Existence Theorem:**
If \( f \in E(0, \infty) \), where \( E(0, \infty) \) is the set of functions that are piecewise continuous on \([0, \infty)\) and of exponential order, then \( \exists \, \alpha \in \mathbb{R} \) such that \( L[f] = \int_0^\infty e^{-st} f(t) \, dt \) exists \( \forall \, s > \alpha \).

Proof:
Definition. The linear transformation $L^{-1}: \text{Rng}(L) \rightarrow V$, where $V$ is the subspace of $E(0, \infty)$ consisting of all continuous functions of exponential order, define by

$$L^{-1}[F](t) = f(t) \iff L[f](s) = F(s)$$

is called the inverse Laplace transform.

Note. A linear transformation is a mapping which preserves addition and scalar multiplication; i.e.,

$$L^{-1}[F + G] = L^{-1}[F] + L^{-1}[G] \quad \text{and} \quad L^{-1}[cF] = cL^{-1}[F].$$

<table>
<thead>
<tr>
<th>Common Inverse Laplace Transforms</th>
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<tbody>
<tr>
<td>Let $n \in \mathbb{N}$, $a, b \in \mathbb{R}$.</td>
</tr>
<tr>
<td>• $L^{-1} \left[ \frac{1}{s^{n+1}} \right] = \frac{1}{n!}t^n$</td>
</tr>
<tr>
<td>• $L^{-1} \left[ \frac{1}{s - a} \right] = e^{at}$</td>
</tr>
</tbody>
</table>

Example 5. Determine the inverse Laplace transform of the given function.

(a) $F(s) = \frac{s + 6}{s^2 + 1}$

(b) $F(s) = \frac{s + 4}{(s - 1)(s + 2)(s - 3)}.$


Definition. A function $f$ defined on an interval $[0, \infty)$ is said to be periodic with period $T$ if $T$ is the smallest positive real number that satisfies the equation

$$f(t + T) = f(t) \quad \forall t \geq 0.$$ 

Theorem 8.3.1. Let $f \in E(0, \infty)$. If $f$ is periodic on $[0, \infty)$ with period $T$, then

$$L[f] = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt.$$ 

Proof:
Example 6. Determine the Laplace transform of the given function.

\[ f(t) = \cos t, \quad 0 \leq t < \pi, \quad f(t + \pi) = f(t). \]
Example 7. Set-up the integral to find the Laplace transform of the given function.

\[ f(t) = |\cos t|, \quad 0 \leq t < \pi, \quad f(t + \pi) = f(t). \]
Theorem 8.4.1. Suppose that \( f \) is of exponential order on \([0, \infty)\) and that \( f' \) exists and is piecewise continuous on \([0, \infty)\). Then \( L[f'] \) exists and is given by \( L[f'] = sL[f] - f(0) \).

Proof:

Example 8. Use Laplace transforms to solve the given initial-value problem.

\[ y' - y = 6 \cos t, \text{ given } y(0) = 2. \]
Basic Steps for solving and Initial-Value Problem using the Laplace Transform

- Take the Laplace transform of the given differential equation, and substitute in the given initial condition.
- Solve the resulting equation algebraically for \( Y(s) \).
- Take the inverse Laplace transform of \( Y(s) \) to determine the solution \( y(t) \) of the given initial-value problem.

**Example 9.** Use Laplace transforms to solve the given initial-value problem.

\[
y'' - y' - 6y = 6(2 - e^t), \text{ given } y(0) = 5 \text{ and } y'(0) = -3.
\]
8.5 The First Shifting Theorem

**Theorem 8.5.1. First Shifting Theorem**

If $L[f] = F(s)$, then

$$L[e^{at} f(t)] = F(s - a).$$

Conversely, if $L^{-1}[F(s)] = f(t)$, then

$$L^{-1}[F(s - a)] = e^{at} f(t).$$

**Proof:**

**Example 10.** Determine $f(t - a)$ for $f(t) = \frac{t}{t^2 + 4}$ and the constant $a = 1$. 


Example 11. Determine \( f(t) \) for \( f(t - 4) = \frac{t + 1}{(t - 1)^2 + 4} \).

Example 12. Determine the Laplace transform of \( f \) for \( f(t) = e^{2t}(1 - \sin^2 t) \).
Example 13. Determine $L^{-1}[F]$ for $F(s) = \frac{2s + 1}{(s - 1)^2(s + 2)}$. 
Example 14. Solve the given initial-value problem.

\[ y'' + y = 5te^{-3t} \text{ given } y(0) = 2 \text{ and } y'(0) = 0. \]
8.6 The Unit Step Function

**Definition.** The unit step function or Heaviside step function, $u_a(t)$, is defined by

$$u_a(t) = \begin{cases} 
0, & 0 \leq t < a, \\
1, & t \geq a,
\end{cases} \text{ where } a \in \mathbb{R}^+.$$  

**Example 15.** Make a sketch of $f(t) = 1 + (t - 1) u_1(t)$ on the interval $[0, \infty)$.

**Example 16.** Make a sketch of the given function on $[0, \infty)$ and express it in terms of the unit step function

$$f(t) = \begin{cases} 
2, & 0 \leq t < 1, \\
2e^{(t-1)}, & t \geq 1
\end{cases}.$$
Example 17. Make a sketch of the given function on $[0, \infty)$ and express it in terms of the unit step function

$$f(t) = \begin{cases} 
0, & 0 \leq t < 2, \\
3 - t, & 2 \leq t < 4, \\
-1, & t \geq 4.
\end{cases}$$
8.7 The Second Shifting Theorem

Theorem 8.7.1. Let \( L[f(t)] = F(s) \). Then

\[
L[u_a(t) f(t - a)] = e^{-as} F(s).
\]

Conversely,

\[
L^{-1}[e^{-as} F(s)] = u_a(t) f(t - a).
\]

Proof:

Corollary 8.7.1. If \( L[f(t)] = F(s) \), then

\[
L[u_a(t) f(t)] = e^{-as} L[f(t + a)].
\]
Example 18. Determine the Laplace transform of the given function

(a) \( f(t) = (\cos t) u_{\pi}(t) \) 
(b) \( f(t) = e^{a(t-c)} \cos[b(t - c)] u_c(t) \) where \( a, b, c \in \mathbb{R}^+ \).

Example 19. Determine the inverse Laplace transform of \( F(s) = \frac{e^{-2s}}{s^2 + 2s + 2} \).
Example 20. Solve $y' - y = 4u_{\pi/4}(t) \cos \left( t - \frac{\pi}{4} \right)$ given $y(0) = 1$. 
8.9 The Convolution Integral

**Definition.** Suppose that $f$ and $g$ are continuous on the interval $[0, b]$. Then for $t \in [0, b]$, the convolution product, $f \ast g$, of $f$ and $g$ is defined by

$$(f \ast g)(t) = \int_0^t f(t - \tau)g(\tau)d\tau.$$ 

The integral

$$\int_0^t f(t - \tau)g(\tau)d\tau$$ 

is called a convolution integral.

**Convolution Integral Properties**

Lef $f$, $g$ be continuous on an interval $I$.

- $f \ast g = g \ast f$
- $f \ast (g \ast h) = (f \ast g) \ast h$
- $f \ast (g + h) = f \ast g + f \ast h$

**Theorem 8.9.1.** If $f$, $g \in E(0, \infty)$, then

$$L[f \ast g] = L[f]L[g].$$

Conversely,

$$L^{-1}[F(s)G(s)] = (f \ast g)(t).$$
Example 21. Determine $f \ast g$ and $L[f \ast g]$ for $f(t) = e^t$ and $g(t) = te^{2t}$.

Example 22. Determine $L^{-1}[F(s)G(s)]$ using the Convolution Theorem for $F(s) = \frac{1}{s^2 + 9}$ and $G(s) = \frac{2}{s^3}$. 
Example 23. Solve $y'' - 2y' + 10y = \cos(2t)$ given $y(0) = 0$ and $y'(0) = 1$ up to the evaluation of the convolution integral.
9 Series Solutions to Linear Differential Equations

9.2 Series Solutions about an Ordinary Point

**Definition.** The point \( x = x_0 \) is called an **ordinary point** of the differential equation
\[
y'' + p(x)y' + q(x)y = 0
\]
if \( p \) and \( q \) are both analytic at \( x = x_0 \). Any point that is not an ordinary point of the differential equation \( y'' + p(x)y' + q(x)y = 0 \) is called a **singular point** of the differential equation.

**Note.** A function is said to be **analytic** at \( x = x_0 \) if it can be represented by a convergent power series centered at \( x = x_0 \) with a nonzero radius of convergence.

*In this section we restrict our attention to ordinary points.*

**Theorem 9.2.1.** Let \( p \) and \( q \) be analytic at \( x = x_0 \), and suppose that their power series expansions are valid for \( |x - x_0| < R \). Then the general solution to the differential equation
\[
y'' + p(x)y' + q(x)y = 0
\]
can be represented as a power series centered at \( x = x_0 \), with radius of convergence at least \( R \). The coefficients in this series can be determined in terms of \( a_0 \) and \( a_1 \) by directly substituting
\[
y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n
\]
into \( y'' + p(x)y' + q(x)y = 0 \). The resulting solution is of the form
\[
y(x) = a_0 y_1(x) + a_1 y_2(x),
\]
where \( y_1 \) and \( y_2 \) are linearly independent solutions to \( y'' + p(x)y' + q(x)y = 0 \) on the interval of existence. If the initial conditions \( y(x_0) = \alpha, \ y'(x_0) = \beta \) are imposed, then \( a_0 = \alpha, a_1 = \beta \).

**Theorem 9.2.2.** If \( p(x) \) and \( q(x) \) are polynomials and \( q(x_0) \neq 0 \), then the power series representation of \( p/q \) has radius of convergence \( R \), where \( R \) is the distance, in the complex plane, from \( x_0 \) to the nearest root of \( q \). Moreover, if \( z = a \pm ib \) is a root of \( q \), then the distance from the center, \( x = x_0 \), of the power series to \( z \) is
\[
|z - z_0| = \sqrt{(a - x_0)^2 + b^2}.
\]

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Example 1. Determine two linearly independent power series solutions to $y'' + xy = 0$ centered at $x = 0$. Also determine the radius of convergence of the series solutions.
Example 2. Determine two linearly independent power series to \( y'' - x^2 y' - 3xy = 0 \) centered at \( x = 0 \). Also, determine the radius of convergence of the series solutions.
Example 3. Determine terms up to and including $x^5$ in two linearly independent power series solutions of $y'' + 2y' + 4xy = 0$. State the radius of convergence of the series solutions.
9.3 The Legendre Equation

**Definition.** \((1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0\) where \(\alpha \in \mathbb{R}\) is called the *Legendre Equation*.

**Example 4.** Determining the series solution and derivation of Legendre Coefficients:
Note. The recurrence relation of the coefficients of the Legendre polynomial is given by
\[ a_{n+2} = \frac{n(n+1) - N(N+1)}{(n+1)(n+2)} a_n \quad n = 0, 1, 2, \ldots \]

Definition. Let \( N \) be a nonnegative integer. The **Legendre polynomial of degree** \( N \), denoted \( P_N(x) \), is defined to be the polynomial solution to
\[(1 - x^2)y'' - 2xy' + N(N+1)y = 0,\]
which has been normalized so that \( P_N(1) = 1 \).

Example 5. Determine the Legendre polynomials \( P_3(x) \) and \( P_4(x) \).
Note. Rodrigue’s Formula:

\[ P_N(x) = \frac{1}{2^N N!} \frac{d^N}{dx^N} (x^2 - 1)^N, \quad N = 0, 1, 2, \ldots \]

Example 6. Determine the Legendre polynomial \( P_4(x) \).

Example 7. Find the first five Legendre polynomials, where \( N \) begins at \( N = 0 \).

Homework. §9.3: T/F #1,3; #2, 3, Find the finite-termed solution to: \((1-x^2)y''-2xy'+30y = 0\).