The spectral decimation of the Laplacian on the Sierpinski gasket

Nishu Lal

University of California, Riverside
Fullerton College

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Construction of the Laplacian $\Delta$ on the Sierpinski gasket

1. Find a sequence of graphs $\Gamma_m$
2. Define the graph Laplacian $\Delta_m$ on each $\Gamma_m$
3. The Laplacian on SG is $\Delta = \lim_{m \to \infty} c_m \Delta_m$

The spectrum of the Laplacian on SG

1. Outline for decimation method
2. The eigenvalue diagram

The spectral zeta function
What is the Sierpinski gasket, SG?
Interesting facts about SG

- The dimension of the gasket is \( \frac{\log 3}{\log 2} \approx 1.5849 \).
- The area of SG is 0.
- The boundary is of infinite length (the circumference is infinity).
- It is connected.
The construction of the Sierpinski gasket

We study calculus on fractals by the approach of using discrete approximations. Consider three contraction mappings for SG:

\[ F_i(x) = \frac{1}{2}(x - q_i) + q_i \]

for \( i = 0, 1, 2 \) where \( \{q_0, q_1, q_2\} \) are the vertices of a triangle.
Let $S(\mathbb{R}^2)$ be the space of nonempty compact subsets of $\mathbb{R}^2$. $S(\mathbb{R}^2)$ is a complete metric space with Hausdorff metric.

**Hausdorff metric**

The Hausdorff metric is a distance function on $S(\mathbb{R}^2)$:

$$d_H(A, B) = \inf\{\epsilon > 0 : B \subseteq A_\epsilon, A \subseteq B_\epsilon\}$$

where $A_\epsilon = \{x \in \mathbb{R}^2 : d(x, A) \leq \epsilon\}$. 


Define $F : S(\mathbb{R}^2) \rightarrow S(\mathbb{R}^2)$ by $F(A) = F_0(A) \cup F_1(A) \cup F_2(A)$. $F$ is a contraction mapping on $S(\mathbb{R}^2)$.

**Definition**

The Sierpinski gasket, $SG \in S(\mathbb{R}^2)$ is a unique fixed point of $F$ or

$$SG = \bigcup_{i=0}^{2} F_i(SG) = F(SG)$$

that is, $SG$ is a self-similar set.
The graph approximation $\Gamma_m \to SG$

The Sierpinski gasket is approximated by a sequence of graphs.

**Definition**

Let $\Gamma_0$ be the graph on $V_0$. Then we define

$$\Gamma_m := \bigcup_{i=0}^{2} F_i(\Gamma_{m-1}).$$
Let \( \Gamma_0 \) be the graph on \( V_0 \) where \( V_0 = \{ q_0, q_1, q_2 \} \). Then the vertices of \( \Gamma_m \) are obtained inductively by

\[
V_m := \bigcup_{i=0}^{2} F_i(V_{m-1}).
\]

Define \( V_* = \bigcup_{m>0} V_m \).

\( V_* \) is dense in SG.

So it is enough to use the sequence of finite sets \( V_m \) to approximate SG.
The graph Laplacian on $\Gamma_m$

**Definition**

Points $x$ and $y$ are called neighbors in $\Gamma_m$ if there is an m-cell containing both or $x \sim_m y$.

**Definition**

Let $u : V_m \rightarrow \mathbb{R}$ be a function. We define the graph Laplacian on $\Gamma_m$ by

$$\Delta_m u(x) := \sum_{x \sim_m y} (u(x) - u(y))$$

where $x \in V_m \setminus V_0$. 
Goal: define $\Delta$ on SG from $\Delta_m$ on $\Gamma_m$

We define the pointwise formulation for the Laplacian $\Delta$ on SG as the renormalized limit of $\Delta_m$

$$\Delta u(x) := \lim_{m \to \infty} c_m \Delta_m u(x)$$

Now the goal is to find the renormalization factor $c_m$. 
We use the concept of graph energy to define the Laplacian.

**Graph Energy on $\Gamma_m \to SG$**

Let $u : V_m\to \mathbb{R}$ be a function. The graph energy of $u$ is

$$E_m(u) = r^{-m} \sum_{x \sim y} (u(x) - u(y))^2$$

where $r = \frac{3}{5}$.

For any continuous function $u$ on SG, $E_m(u|_{V_m})$ is a monotone increasing sequence of $m$.

**Graph energy on SG**

$$E(u) = \lim_{m \to \infty} r^{-m} \sum_{x \sim y} (u(x) - u(y))^2$$
To explore the relationship between $\mathcal{E}_m(u)$ and $\mathcal{E}_{m+1}(u)$, we consider the following:

- Given $u := V_m \rightarrow \mathbb{R}$, extend $u$ to $V_{m+1}$.
- There are many extensions, but there is one unique extension $\tilde{u}$ that minimizes $\mathcal{E}_{m+1}(u)$.
- $\tilde{u}$ is called the harmonic extension.
- The harmonic extension satisfies $\frac{1}{5} - \frac{2}{5}$ rule at the new points $V_{m+1} \setminus V_m$.
- Then we get $\mathcal{E}_{m+1}(\tilde{u}) = \mathcal{E}_m(u)$ for $\forall u$. 
Formulation of the Laplacian on SG

To define the Laplacian from the graph energy requires the choice of a self-similar measure $\mu$.

Weak formulation of the Laplacian $\Delta_\mu$

Let $u \in \text{dom} \mathcal{E}$ and $f \in C(SG)$. Then $u \in \text{dom} \Delta_\mu$ and $\Delta_\mu u = f$ if

$$\mathcal{E}(u, v) = - \int_{SG} f v d\mu$$

$\forall \, v \in \text{dom} \mathcal{E}$ vanishing on the boundary points.
We can derive the pointwise formulation of the Laplacian from the weak formulation.

- The Laplacian on SG is

\[
\Delta_\mu u(x) = \frac{3}{2} \lim_{m \to \infty} 5^m \Delta_m u(x).
\]
The normalized discrete Laplacian

From now on, we will discuss the normalized discrete Laplacian. It is defined as follows: for $u : V_m \to \mathbb{R}$ with $u$ vanishing on $V_0$, the normalized Laplacian on $\Gamma_m$ is

$$\Delta_m u(x) := \frac{1}{4} \sum_{x \sim y} (u(x) - u(y))$$

where $x \in V_m \setminus V_0$. 
Obtain solutions of the eigenvalue equation

$$-\Delta u = \lambda u$$

on SG as limits of solutions of the discrete version

$$-\Delta_m u_m = \lambda_m u_m$$
on $V_m \setminus V_0$.

Construct $\lambda_{m+1}$ from $\lambda_m$ via a polynomial $R(z)$. 
Outline of the decimation method

- If \( u \) is an eigenfunction of \(-\Delta_{m+1}\) with eigenvalue \( \lambda \), that is, \(-\Delta_{m+1}u = \lambda u \) and \( \lambda \notin B \), then
  \[
  -\Delta_m(u|_{V_m}) = R(\lambda)u|_{V_m}
  \]
  where \( B = \{\frac{5}{4}, \frac{1}{2}, \frac{3}{2}\} \) is the set of forbidden eigenvalues.

- If \(-\Delta_m u = R(\lambda)u \) and \( \lambda \notin B \) then there exists a unique extension \( \tilde{u} \) of \( u \) such that
  \[
  -\Delta_{m+1}(\tilde{u}) = \lambda(\tilde{u}).
  \]
The special polynomial $R(z)$

- The decimation method relates the eigenvalues of successive graph Laplacians by the polynomial $R(z) = z(5 - 4z)$.
- Given the eigenvalues of $-\Delta_0$ to be $\{0, \frac{3}{2}\}$. To obtain eigenvalues of $-\Delta_1$, we consider the inverse images of $R(z)$:
  - $R_-(z) = \frac{5 - \sqrt{25-16z}}{8}$.
  - $R_+(z) = \frac{5 + \sqrt{25-16z}}{8}$.
The eigenvalue diagram

<table>
<thead>
<tr>
<th>Operators</th>
<th>Eigenvalues</th>
</tr>
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<tbody>
<tr>
<td>$-\Delta_0$</td>
<td>0 [1]</td>
</tr>
<tr>
<td>$-\Delta_1$</td>
<td>$\frac{5}{4}$ [1]</td>
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Eigenvalues of the Laplacian $\Delta_\mu$

Every eigenvalue of $\Delta_\mu$ has a form

$$5^i \lim_{m \to \infty} 5^m (R_-)^{m-j} R_\omega(z_0)$$

where $\omega$ is a word with $|\omega| = j$ taking values in $\{-, +\}$ and $(R_-)^{m-j}$ is the $(m-j)$th iteration power of $R_-$ and $z_0 = \frac{3}{4}, \frac{5}{4}$.

- If $R_-(z) < z$ then $R_-^n(z)$ is decreasing so it converges to a fixed point of $R_-$ which is 0.
- The limit exists if $R_-$ is applied after finite steps.
The spectral zeta function

Definition

Suppose $A$ is a non-negative self-adjoint operator with discrete spectrum and positive eigenvalues $\{\lambda_j\}$, then its spectral zeta function is

$$\zeta_A(s) = \sum_{\lambda_j \neq 0} (\lambda_j)^{-s/2}. $$

Example

Consider the Dirichlet Laplacian on $[0,1]$, $A = -\frac{d^2}{dx^2}$.

Eigenvalues: $\lambda = \pi^2 j^2$, $j = 1, 2, ...$

The spectral zeta function is

$$\zeta_A(s) = \sum_{j=1}^{\infty} (\pi^2 j^2)^{-s/2}$$

$$= \pi^{-s} \sum_{j=1}^{\infty} j^{-s}$$

$$= \pi^{-s/2} \zeta(s).$$
Alexander Teplyaev discovered the product structure of the spectral zeta function of the Laplacian on SG that involves a geometric part and a new zeta function associated with the polynomial $R(z)$. This zeta function has many interesting properties...

A similar product structure of the spectral zeta function was studied earlier by Michel Lapidus for fractal strings and can now be viewed as a special case of this situation.
A fractal string $\mathcal{L}$ is a disjoint collection of open intervals of length $\ell_k$. Define its geometric zeta function to be

$$\zeta_{\mathcal{L}}(s) = \sum_{k=1}^{\infty} (\lambda_k)^s.$$ 

Let $\mathcal{L} = \bigcup I_k$ with $|I_k| = \ell_k$ and let $A = \frac{-d^2}{dx^2}$ be the Dirichlet Laplacian on $\mathcal{L}$. $\sigma(A) = \bigcup_{k=1}^{\infty} \sigma(A; I_k) = \{ \frac{\pi^2 n^2}{\ell_k^2} : n \geq 1, k \geq 1 \}$. The spectral zeta function of $A$:

$$\zeta_A(s) = \sum_{k,n \geq 1} \left( \frac{\pi^2 n^2}{\ell_k^2} \right)^{-s/2} = \pi^{-s} \sum_{k=1}^{\infty} \ell_k^s \sum_{n=1}^{\infty} n^{-s}.$$ 

### Theorem

Lapidus: $\zeta_A(s) = \pi^{-s} \zeta_{\mathcal{L}}(s) \zeta(s)$, where $\zeta(s)$ is the Riemann zeta function.
Current research

- In order to generalize the decimation method for a certain class of fractals, Sabot introduced a function of several complex variables called the renormalization map.

- It is interesting to study the spectral zeta function of other self-similar sets in similar setting, in particular, its product structure in terms of the zeta function associated with the renormalization map.

- Thus far, jointly with Michel Lapidus, we have looked at the Sturm-Liouville operator of the form \( \frac{d}{d\mu} \frac{d}{dx} \) on the half real line (blow-up of a unit interval). We established the spectral zeta function in terms of a renormalization map induced from the decimation method discovered by Sabot. The dynamics of an invariant curve of the renormalization map is used to compute the spectrum of the operator. We define the zeta function \( \zeta_f \) associated with the renormalization map \( f \) and relate it to the spectral zeta function \( \zeta_{sp} \) of the Sturm-Liouville operator via a product formula with a hyperfunction.


