INTRODUCTION TO TROPICAL MATHEMATICS
Objectives

- Origin of “Tropical” Mathematics
- Applications
- Definitions and Arithmetic operations
- Monomials
- The Freshman Dream
- Possible research project
Origin of “Tropical” Mathematics
Origin of “Tropical” Mathematics
Applications of Tropical Mathematics

- Geometric Combinatorics
- Algebraic Geometry
- Mathematical Physics
- Number Theory
- Symplectic Geometry
- Computational Biology
Tropical Mathematics

Tropical mathematics is the study of the tropical semiring:

\[(\mathbb{R} \cup \{\infty\}, \oplus, \odot)\]
**Ring Definition**

**Ring:** An algebraic structure consisting of:

- A set
  - Together with two binary operations (usually addition and multiplication)

With the following axioms:
- Addition is commutative
- Addition and multiplication are associative
- Multiplication distributes over addition
- There exists an additive identity
- Each element in the set has an *additive inverse*
Semiring Definition

Semiring:

Similar to a ring, but without the requirement that each element must have an additive inverse.
Arithmetic operations

Tropical sum: \((\mathbb{R} \cup \{\infty\}, \oplus, \odot)\)

\(\oplus\) operator \hspace{1cm} // “tropical sum” operator

\[ x \oplus y := \min\{x, y\} \]

\[ 1 \oplus 2 = 1 \]
\[ 5 \oplus 8 = 5 \]
Arithmetic operations

Tropical product:

\( (\mathbb{R} \cup \{\infty\}, \oplus, \odot) \)

\( \odot \) operator

// “tropical product” operator

\[ x \odot y := x + y \]

\[ 1 \odot 2 = 3 \]
\[ 5 \odot 8 = 13 \]
Both addition and multiplication are commutative.

\[
x \oplus y = y \oplus x
\]
\[
4 \oplus 7 = 7 \oplus 4
\]
\[
4 = 4
\]

\[
x \circ y = y \circ x
\]
\[
3 \circ 8 = 8 \circ 3
\]
\[
11 = 11
\]
Arithmetic operations

Order of Operations is the same.

Distributive Law holds for both operations.

\[ x \circ (y \oplus z) = x \circ y \oplus x \circ z \]

\[ 4 \circ (3 \oplus 7) = 4 \circ 3 \oplus 4 \circ 7 \]
   \[ = 7 \oplus 7 \]
   \[ = 7 \]
Arithmetic operations

Neutral Elements: \((\mathbb{R} \cup \{\infty\}, \oplus, \odot)\)

\[
\begin{align*}
    x + 0 &= x \\
    x \cdot 1 &= x \\
    x \oplus \infty &= x \\
    4 \oplus \infty &= 4 \\
    x \odot 0 &= x \\
    4 \odot 0 &= 4
\end{align*}
\]

\(\infty\) is the additive identity.

\(\infty\) is the additive identity.

// remember we’re adding here
Additive Inverse:

The additive inverse, or opposite, of a number \( a \) is the number that, when added to \( a \), yields zero.

Additive inverse of 7 is \(-7\),
\[ 7 + (-7) = 0, \]

The additive inverse of a number is the number's negative.
Additive Inverse:

$$\begin{align*}
7 + (-7) &= 0 \\
7 \oplus (-7) &= -7 & // \min\{7, -7\}
\end{align*}$$

Additive inverse does not exist for the tropical semiring.
Arithmetic operations

Tropical difference:

11 ⊕ 3 = ?

This really means:

3 ⊕ x = ?

Change to: 11 ⊕ -3 = -3

(ℝ ∪ {∞}, ⊕, ⊙)

// impossible.

// min{3, x}; no solution
// x is undefined

// illegal;
// no additive inverse
Tropical Semiring Definition

Tropical Semiring: \((\mathbb{R} \cup \{\infty\}, \oplus, \odot)\) An algebraic structure consisting of:

- A set \(\mathbb{R}\) ✓
- Together with two binary operations \(\oplus, \odot\) ✓
- With the following axioms:
  - Addition is commutative, ✓
  - Addition and multiplication are associative, ✓
  - Multiplication distributes over addition, ✓
  - There exists an additive identity ✓
  - No additive inverse ✓
Monomials

Let \( x_1, x_2, x_3, \ldots, x_n \) be variables that represent elements in the tropical semiring \( (\mathbb{R} \cup \{\infty\}, \oplus, \odot) \).

A monomial is any product of these variables, where repetition is allowed.

A monomial represents a function from \( \mathbb{R}^n \) to \( \mathbb{R} \).

\[
x_2 \odot x_1 \odot x_3 \odot x_1 \odot x_4 \odot x_2 \odot x_3 \odot x_2 = x_1^2 x_2^3 x_3^2 x_4
\]

\[
x_2 + x_1 + x_3 + x_1 + x_4 + x_2 + x_3 + x_2 = 2x_1 + 3x_2 + 2x_3 + x_4.
\]

// = linear function

\( \therefore \) Tropical monomials are linear functions with integer coefficients.
The Freshman Dream

Definition:

\[(x + y)^3 = x^3 + y^3\] //not true

\[(x + y)^3 = x^3 + x^2y + xy^2 + y^3\] //true

But! Freshman Dream holds for Tropical Arithmetic

\[(x \oplus y)^3 = x^3 \oplus y^3\] //true
The Freshman Dream

Proof:

\[(x \oplus y)^3 = (x \oplus y) \odot (x \oplus y) \odot (x \oplus y)\]

\[= (x \odot x \odot x) \oplus (x \odot x \odot y) \oplus (x \odot y \odot y) \oplus (y \odot y \odot y)\]

\[= x^3 \oplus x^2 y \oplus xy^2 \oplus y^3\]

\[= x^3 \oplus y^3\]
The Freshman Dream

Example:

\[(3 \oplus 2)^3 = 3^3 \oplus 2^3\]

\[= (3 \oplus 2) \odot (3 \oplus 2) \odot (3 \oplus 2)\]

\[= (3 \odot 3 \odot 3) \oplus (3 \odot 3 \odot 2) \oplus (3 \odot 2 \odot 2) \oplus (2 \odot 2 \odot 2)\]

\[= 3^3 \oplus 3^2 \odot 2 \oplus 3 \odot 2^2 \oplus 2^3\]

\[= 3^3 \oplus 2^3\]

\[= 2^3\]

\[= 6\]
Research Problem

Research problem:

The tropical semiring generalizes to higher dimensions: The set of convex polyhedra in $\mathbb{R}^n$ can be made into a semiring by taking $\odot$ as “Minkowski sum” and $\oplus$ as “convex hull of the union.” A natural subalgebra is the set of all polyhedra that have a fixed recession cone $C$. If $n = 1$ and $C = \mathbb{R}_{\geq 0}$, this is the tropical semiring.

Develop linear algebra and algebraic geometry over these semirings, and implement efficient software for doing arithmetic with polyhedra when $n \geq 2$. 