RESEARCH PROGRAM SUMMARY - DANA D. CLAHANE

1. Introduction

My research currently consists of three major components, which are respectively delineated in the following three sections. These three components are (1) continued work on weighted and unweighted composition operators, (2) a delving into related questions in holomorphic function theory, including fixed point sets and Denjoy-Wolff theorems on general domains, and (3) a broadening component involving study of Hankel operators, free probability, and imaging.

Composition operators have arisen in de Brange's original proof of the Bieberbach conjecture [De] and are known to have applications to control (cf. [DrStep], [Ry]). Free probability was pioneered by Voiculescu in an effort to handle the isomorphism problem for free group factors and has intersections with the representation theory of symmetric groups and with recent work [CanRoT] on image/signal construction from sparse/inaccurate data. Analytical issues on the interface between imaging science and mathematics are highlighted by progress on the boundary rigidity problem and its relation to use of propagated seismic waves to study the structure of the Earth's core [StefU]. Applications of time-frequency analysis include radar imaging/signal noise removal [CheVL] and dynamics of the circular restricted three-body problem in celestial mechanics [AreMar].

2. Function-theoretic Operator Theory

Suppose that $\mathcal{X}(D)$ is a Banach space of holomorphic functions that are defined on a domain D in $\mathbb{C}^n$, and let $\phi : D \to D$ and $\psi : D \to \mathbb{C}$ be holomorphic. Let $\mathcal{H}(D)$ denote the vector space of holomorphic functions on D. We define the *weighted composition operator* $W_{\psi,\phi} : \mathcal{X}(D) \to \mathcal{H}(D)$ by $W_{\psi,\phi}(f) = \psi(f \circ \phi)$. $W_{1,\phi}$ is simply called a *composition operator* and is abbreviated by C_ϕ. Two common domains that play the role of D in our work are the *unit ball* $\mathbb{B}_n := \{z \in \mathbb{C}^n : |z| < 1\}$ and the *unit polydisk* $\Delta^n := \{z = (z_1, z_2, \ldots z_n) \in \mathbb{C}^n : z_i \in \mathbb{B}_1 \ \forall i\}$.

Any bounded symmetric domain D is identified with a biholomorphically equivalent, balanced, and convex domain containing the origin) and has a Shilôv boundary, which we denote by S. Let $p > 0$. Two holomorphic function spaces on which the action of composition operators is often studied are the Hardy space $H^p(D)$, which consists of those holomorphic functions f on D whose boundary-value functions on D are in $L^p(S)$, with "norm" given by the usual L^p-integral of the boundary-value functions on S, and the Bergman space, which is the space of holomorphic functions in $L^p(D)$, with "norm" given by the L^p-integrals over D. Similar definitions of these spaces exist for other types of domains.

C_ϕ is always bounded on $H^p(\mathbb{B}_n)$ and $A^p_\alpha(\mathbb{B}_n)$ when $n = 1$ [CoMac, Ch. 2]; however, this fact does not hold for $n > 1$ (cf. [CiW]). Compactness is even less well-understood in several variables. To date, purely function-theoretic characterizations of mappings that induce bounded or compact composition operators on many function spaces in several variables have been quite elusive (see [Rus1] for a survey). In [Cl1], I obtained a purely function-theoretic, sufficient condition for compactness of C_ϕ on $A^p_\alpha(\mathbb{B}_n)$.

One strategy that others and I have pursued is to consider these problems on closely related function spaces (cf. [Mad]). For example, I will extend a well-known result in [Mad] to $\mathbb{B}_n$ by characterizing the holomorphic self-maps of $\mathbb{B}_n$ that induce bounded composition operators on the holomorphic generalized α-Bloch and α-Lipschitz spaces. I solved this problem for Δ^n in [Cl3] and extended it further with Stević/Zhou in [ClStevZho].

Letting $\alpha \in (0,1]$ and $D \subset \mathbb{C}^n$ be a domain, we denote by $\mathcal{L}_\alpha(D)$ the *holomorphic α-Lipschitz space of* D, which is the linear space of $f \in \mathcal{H}(D)$ such that for some $C \geq 0$, $|f(z) - f(w)| \leq C|z-w|^\alpha$ for all $z, w \in D$. For $\alpha \in (0, \infty)$ $\mathcal{B}^\alpha(\Delta^n)$ denotes the *α-Bloch space of* Δ^n, which is the linear space of all $f \in \mathcal{H}(\Delta^n)$ satisfying

$$\sup_{z \in \Delta^n} \left| \frac{\partial f}{\partial z_j}(z) \right| (1 - |z_j|^2)^\alpha < \infty \quad \text{for all } j.$$

Similar definitions can be made on $\mathbb{B}_n$. In [ClStev], S. Stević and I obtained necessary conditions and sufficient conditions for boundedness of C_ϕ on $\mathcal{L}_\alpha(\mathbb{B}_n)$. I will continue searching for a single function-theoretic condition that is both necessary and sufficient, and I will also consider analogous results on vector-valued function spaces (cf. [LiuSakTy]).

In [Sh1], it was shown that compactness of C_ϕ on automorphism-invariant, boundary-regular small spaces of $\mathbb{B}_n$ implies that $||\phi||_\infty < 1$ (see [CoMac, Ch. 4]). The converse of this fact holds for many spaces on $\mathbb{B}_n$ if it is additionally assumed that ϕ is in the given space. I plan to obtain analogues of these results for Δ^n, which is interestingly harder to handle than $\mathbb{B}_n$. I also plan to study C_ϕ when ϕ is not a self-map (see [ZhuStes]). Boundedness/compactness of weighted/unweighted composition operators on more general weighted Sobolev spaces of Δ^n and $\mathbb{B}_n$ (cf. [ChoKSm]) will also be studied.

In another direction, I will also study operators on Hilbert spaces of Dirichlet series, which are related to the Riemann-zeta function (cf. [Mc], [FQVol]). With M. Lapidus, I will explore connections between these ideas and emerging concepts in fractal geometry. I will also delve into function-theoretic operator theory on reproducing kernel spaces of so-called "hyperholomorphic functions", which are quaternionic analogues of classical holomorphic functions [AlMShVol].

I will consider the following one-variable question from [ShSm]: Suppose that G is a proper, simply connected subset of $\mathbb{C}$, and let τ be the Riemann map of Δ onto G. For $\omega \in \partial\Delta$ and $z \in G$, let $\phi_w(z) = \tau[w\{\tau^{-1}(z)\}]$. Let $E(G)$ be the set of all w such that C_{ϕ_w} is bounded on the Hardy space $H^2(G)$ arising from τ. If $n \geq 3$ and Γ_n is a subgroup of order n in $\partial\Delta$, then Γ_n must be the subgroup of nth roots of unity. We are then asked: "Can an infinite subgroup of $\partial\Delta$ be equal to $E(G)$ for some simply connected domain G that is properly contained in $\mathbb{C}$?".

In the recent paper [GhZheZo], necessary and sufficient conditions for C_ϕ to have closed range on $\mathcal{B}^1(\Delta)$ were given in terms of "sampling sets". After [GhZheZo], $H \subset \Delta$ is called a *sampling set* for $\mathcal{B}^1(\Delta)$ iff there is a $C > 0$ such that for all $f \in \mathcal{B}^1(\Delta)$,

$$\sup_{z \in \Delta}\{(1 - |z|^2)|f'(z)|\} \leq C \sup_{z \in H}\{(1 - |z|^2)|f'(z)|\}.$$

I will pursue generalizations of these ideas to α-Bloch spaces, even in higher dimensional domains such as Δ^n, $\mathbb{B}_n$, and more general domains.

Comparison results will also be considered, such as the fact that boundedness (respectively, compactness) of a composition operator on a Hardy space of a bounded symmetric domain implies boundedness (respectively, compactness) of the operator on the weighted Bergman spaces with same exponent. This result in the case of $\mathbb{B}_n$ appears in my first paper [Cl1]. Comparison theorems such as these should hold for other weighted Sobolev spaces of $\mathbb{B}_n$ and Δ^n and are of foundational importance.

The classical Denjoy-Wolff theorem says that the iterates of a fixed-point free analytic self-map of Δ converge uniformly on compact subsets to a unique point, called the "Denjoy-Wolff" point of ϕ, on $\partial\Delta$. This theorem was extended to $\mathbb{B}_n$ in [Mac1] and used in [Mac2] to show that the spectrum of a compact or power-compact composition operator on $H^p(\mathbb{B}_n)$ for $p \geq 1$ consists of $0, 1$, and all possible products of eigenvalues of $\phi'(z_0)$, where z_0 is what she showed to be the unique fixed point of ϕ, thus extending a well-known result [CauSc] for $n = 1$. I generalized these results to more general bounded symmetric domains where the Denjoy-Wolff theorem is known to fail [ChuMe], including Δ^n in [Cl2]. A corollary of my result is that the spectrum result of [Mac1] holds for all $H^p(\Delta^n)$ with $p > 1$ and $A^p_\alpha(\Delta^n)$ for $p > 0$. I will address the question of what happens for $H^p(D)$ when $p \neq 2$ for general bounded symmetric domains or when $p \in (0, 1]$ in the case of Δ^n.

In the recent paper [Cl4], I proposed the following statement:

Conjecture: *Suppose that $D \subset \mathbb{C}^n$ is a bounded, convex domain such that a given Hilbert space of holomorphic functions $\mathcal{H}$ (on D) in which the polynomials are contained densely has reproducing kernel K satisfying $K(z, z) \to \infty$ as $z \to \partial D$. Let $\psi : D \to \mathbb{C}$ be holomorphic, and suppose that ψ is bounded away from 0 toward ∂D. Assume that $\phi : D \to D$ is holomorphic and that $W_{\psi,\phi}$ is compact on $\mathcal{H}$. Then*

(1) ϕ has a unique fixed point in D, and

(2) the spectrum of $W_{\psi,\phi}$ is the set $\{\psi(a)\sigma : \sigma \in E\}$, where E is the set consisting of 0, 1, and all possible products of eigenvalues of $\phi'(a)$.

In [Cl4], I made initial progress by proving part (1) of the conjecture. Part (2) of the conjecture remains open, even for the Hardy and weighted Bergman spaces of either $\mathbb{B}_n$ or Δ^n, so I will search for resolution, which would extend a recent result [Gu] in the special case when $D = \Delta$ and $\mathcal{H}$ is the weighted Hardy space of analytic functions $f : \Delta \to \mathbb{C}$ whose MacClaurin series $f(z) = \sum_{j=0}^{\infty} a_j z^j$ satisfy $\sum_{j=0}^{\infty} |a_j|^2 b_j^2 < \infty$, where $(b_j)_{j\in\mathbb{N}}$ is a sequence of positive numbers such that $\liminf_{j\to\infty} b_j^{1/j} \geq 1$.

3. Some Related Questions in Holomorphic Function Theory

In the second component of this research program, we will study fixed point sets of holomorphic maps. In [Hu], it was shown that the iterates of a holomorphic self-map of a topologically contractible, strictly pseudoconvex domain in $\mathbb{C}^n$ form a compactly divergent sequence; however, in [AbH], M. Abate/P. Heinzner showed the existence of a holomorphic self-map ϕ of a topologically contractible *pseudoconvex* domain such that the iterates of ϕ do not compactly diverge. I will seek extensions of these results, and I will prove results about the various dimensions (Hausdorff, etc.) of the fixed point set under these hypotheses.

Given a domain $D \subset \mathbb{C}^n$ and a holomorphic self-map $\phi : D \subset D$, can ϕ have finitely many fixed points and more than one fixed point? If we allow the domain to be non-contractible, then the answer can be "Yes!", while the answer for convex domains is known to be "No". I will therefore address the question of whether or not a holomorphic self-map of a pseudoconvex domain in $\mathbb{C}^n$ can have more than one but finitely many fixed points.

If D is a given bounded domain in $\mathbb{C}^n$ and $k \in \mathbb{N}$, $k \neq 1$ is given, is there a holomorphic self-map of D with precisely k fixed points? It is known [Vi] that if a holomorphic self-map of such a domain D does have precisely k fixed points for some $k > 1$, then D cannot be convex. Also, an analytic self-map of a bounded domain in $\mathbb{C}$ with three distinct fixed points must be the identity mapping [K, Thm. 1.7.6]. It is therefore natural to prove or disprove that if $D \subset \mathbb{C}^n$ is a domain and $\phi : D \to D$ is a holomorphic map that fixes $2^n + 1$ points in D, then ϕ must be the identity.

I will also focus on the fundamental problem of obtaining an analogue of the Denjoy-Wolff theorem for irreducible bounded symmetric domains, since this theorem fails in Δ^n for $n = 2$ (see [ChuMe]).

4. New areas: Free Probability, Imaging, and Generalized Hankel operators

Suppose that μ is a compactly supported measure on $\mathbb{R}$ with nonzero variance. If we denote by μ^n the n-fold free additive convolution of μ with itself, then μ^n is absolutely continuous after finitely many n; furthermore, the density of μ^n eventually looks like a semicircle [BerVo2]. It was suggested to me by H. Bercovici that a good initial project in this area would be to extend the above facts about μ^n to the case of unbounded supports, or even to the case of finite moments (see [BerVo2] and [BerVo3]). I will also investigate the regularity properties of μ^n for these sufficiently large values of n near the left- and right- hand boundaries of the semi-circle. I will study the special case of random matrices, which are connected to imaging; in fact, I also plan to consider the special case of random matrices, which are connected to imaging science. My first interdisciplinary project in imaging will consist of collaboration with U. of Minnesota Astronomy Professors R. Humphreys and T. Jones on visualization of the 3D morphology of VY CMa (VY Canis Majoris), which is one the most quickly evolving, luminous and largest stars known (radius 2 x 10^8 km, or larger than Saturn's orbit).

I will also expand my research to operators besides composition operators: If Ω is either a strictly pseudoconvex domain with smooth boundary or a bounded symmetric domain in $\mathbb{C}^n$ and X is for example the Bergman space $A^2(\Omega)$, then one can define a *Hankel operator* H_b by $H_b(f) = [(I - P)M_{\overline{b}}P](f)$ for $f \in L^2(\Omega)$, where b is a holomorphic function on Ω, $P : L^2(\Omega) \to A^2(\Omega)$ is the orthogonal projection, and I is the identity operator [Ara].

Generalized Hankel operators $\mathcal{A}_b$ for holomorphic functions b on Cartan domains, the generalized Bloch and little Bloch spaces $\mathcal{B}$ and $\mathcal{B}_0$, and the generalized $\mathcal{B}MOA$ and $\mathcal{V}MOA$ spaces on these domains were introduced in [Ara], in which it is shown that $\mathcal{A}_b$ is bounded on the weighted Bergman space D of a Cartan domain of tube type with rank $r > 1$ iff $b \in \mathcal{B}$ iff $b \in \mathcal{B}MOA$ and that $\mathcal{A}_b$ is compact iff $b \in \mathcal{B}_0$ iff $b \in \mathcal{V}MOA$. These results extend work appearing in [Ax], [AraFisPe], [BekBerCobZhu] and [Zhu1]. The introduction to [Ara] poses the problem of extending the above results in the case of Cartan domains of tube type to general Cartan domains in $\mathbb{C}^n$, so I will expand my knowledge of analysis on these domains by extending these results.

References

[AbH] M. Abate/P. Heinzner. *Holomorphic actions on contractible domains without fixed points*, Math. Z. **211** (1992), no. 4, 547-555.

[AlMShVol] D. Alpay/M. Shapiro/D. Volok. *Rational hyperholomorphic functions in* $\mathbb{R}^4$, J. Functional Anal. **221** (2005), 122-149.

[Ara] J. Arazy. *Boundedness and compactness of generalized Hankel operators on bounded symmetric domains*, J. Funct. Anal. **137** (1996), 97-151.

[AraFisPe] J. Arazy/S. Fisher/J. Peetre. *Hankel operators on weighted Bergman spaces*, Amer. J. Math. **110** (1988), 989-1054.

[AreMar] L. V. Vela-Arevalo/J. E. Marsden. *Time-frequency analysis of the restricted three-body problem: transport and resonance transitions*, Classical Quantum Gravity **21** (2004), no. 3, S351-S375.

[Ax] S. Axler. *The Bergman space, the Bloch space, and commutators of multiplication operators*, Duke Math J. **53** (1986), 315-332.

[BeaBu] F. Beatrous/J. Burbea. *Holomorphic Sobolev spaces of the unit ball*, Dissertationes Math. (Rozprawy Mat.), **276**, (1989), 1-57.

[BekBerCobZhu] D. Bekollé/Charles Berger/Lewis Coburn/Kehe Zhu. *BMO in the Bergman metric on bounded symmetric domains*, J. Funct. Anal. **93** (1990), 310-350.

[BerVo2] H. Bercovici/D. Voiculescu. *Superconvergence to the central limit and failure of the Cramér theorem for free random variables*, Probab. Theory Related Fields **103** (1995), no. 2, 215-222.

[BerVo3] H. Bercovici/D. Voiculescu. *Regularity questions for free convolution*, Nonselfadjoint operator algebras, operator theory, and related topics, 37-47, Oper. Theory Adv. Appl., **104**, Birkhuser, Basel, 1998.

[CanRoT] E. Candes/J. Romberg/T. Tao. *Robust uncertainty principles: signal reconstruction using incomplete and incomplete data*, preprint, 2004.

[CauSc] J. Caughran/H. Schwartz. *Spectra of compact composition operators*, Proc. Amer. Math. Soc., **51**, 1975, 127-130.

[CheVL] V. C. Chen/H. Ling. *Time-frequency transforms for radar imaging and signal analysis,* Artech House, Inc., Boston, MA, 2002.

[ChoKSm] B. R. Choe/H. Koo/W. Smith. *Composition operators acting on holomorphic Sobolev spaces*, Trans. Amer. Math. Soc. **355** (2003), no. 7, 2829-2855.

[ChuMe] C. H. Chu/P. Mellon. *Iteration of compact holomorphic maps on a Hilbert ball*, Proc. Amer. Math. Soc. **125** (1997), no. 6, 1771-1777.

[CiW] J. A. Cima/W. R. Wogen. *Unbounded composition operators on* $H^2(B_2)$, Proc. Amer. Math. Soc., **99** (1987), no. 3, 477-483.

[Cl1] D. D. Clahane. *Compact composition operators on weighted Bergman spaces of the unit ball*, J. Operator Theory, **45**, 2001, 335-355.

[Cl2] D. D. Clahane. *Spectra of compact composition operators over bounded symmetric domains*, Integral Equations Operator Theory **51** (2005), no. 1, 41-56.

[Cl3] D. D. Clahane. *Bounded composition operators on Lipschitz and Bloch spaces of the polydisk*, Integral Equations Operator Theory, accepted.

[Cl4] D. D. Clahane. *Compact weighted composition operators and fixed points in convex domains*, Fixed Pt. Theory Appl., submitted, 2005.

[Cl5] D. D. Clahane. *Composition operators on small spaces of the polydisk*, in preparation.

[ClStev] D. D. Clahane/S. Stević. *Norm equivalence and composition operators between Bloch/Lipschitz spaces of the ball*, J. Inequalities Appl., to appear.

[ClStevZho] D. D. Clahane/S. Stević/Z. Zhou. *Composition operators between generalized Bloch spaces of the polydisk*, Integral Equations Operator Theory, submitted, 2005.

[CoMac] C. C. Cowen/B. D. MacCluer. *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Roton, 1995.

[De] L. de Branges. *A proof of the Bieberbach conjecture*, Acta Math., **154** (1985), no. 1-2, 137-152.

[DrStep] Drakhlin/Stepanov M. E. Drakhlin/Eugene Stepanov. *Convergence of composition operators and optimal control problems*, Proceedings of the Second World Congress of Nonlinear Analysts, Part 1 (Athens, 1996), Nonlinear Anal. **30** (1997), no. 1, 505-512.

[FQVol] C. Finet/H. Queffélec/A. Volberg. *Compactness of composition operators on a Hilbert space of Dirichlet series*, J. Funct. Anal. **211** (2004), 271-287.

[GhZheZo] P. Ghatage/D. Zheng/N. Zorboska. *Sampling sets and closed range composition operators on the Bloch space*, Proc. Amer. Math. Soc. **133** (2004), 1371-1377.

[Gu] G. Gunatillake. *Spectrum of a compact weighted composition operator*, Proc. Amer. Math. Soc., to appear.

[Hu] X. Huang. *A non-degeneracy property of extremal mappings and iterates of holomorphic self-mappings*, Ann. Scuola Norm. Pisa Cl. Sci. (4) **21** (1994), no. 3, 399-413.

[K] S. G. Krantz. *Geometric Analysis and Function Spaces*, CBMS Regional Conference Series in Mathematics, **81**, Amer. Math. Soc., Providence, 1993.

[LiuSakTy] Liu, P./Saksman, E./ Tylli, H. O.. *Small composition operators on analytic vector-valued function spaces*, Pacific J. Math., **184** (1998), no. 2, 295-309.

[Mac1] B. D. MacCluer. *Iterates of holomorphic self-maps of the unit ball in* $\mathbb{C}^n$, Mich. Math. J. **30**, (1983), 97-106.

[Mac2] B. D. MacCluer. *Spectra of compact composition operators on* $H^p(\mathbb{B}_n)$, Michigan Math J., **32**, (1985), 237-248.

[Mad] K. M. Madigan. *Composition operators on analytic Lipschitz spaces*, Proc. Amer. Math. Soc. **119**, (1993), no. 2, 465-473.

[Mc] J. McCarthy. *Hilbert spaces of Dirichlet series and their multipliers*, Trans. Amer. Math. Soc. **356** (2004), no. 3, 881-893.

[Rus1] B. Russo. *Holomorphic composition operators in several complex variables*, Contemp. Math., 213, *Amer. Math. Soc.*, Providence 1998.

[Ry] V. V. Ryzhikov. *Stochastic intertwinings and multiple mixing of dynamic systems*, J. Dynam. Control Systems **2** (1996), no. 1, 1-19.

[Sh1] J. H. Shapiro. *Compact composition operators on spaces of boundary-regular holomorphic functions*, Proc. Amer. Math. Soc., **100** (1987), no. 1, 49-57.

[ShSm] J. H. Shapiro/W. Smith. *Hardy spaces that support no compact composition operators*, J. Funct. Anal. **205** (2003), no. 1, 62-89.

[StefU] P. Stefanov/G. Uhlmann. *Boundary rigidity and stability for generic simple metrics*, J. Amer. Math. Soc. **18** (2005), no. 4, 975-1003.

[Vi] J.-P. Vigué. *Points fixes d'applications holomorphes dans un domaine born convexe de* C^n [Fixed points of holomorphic mappings in a bounded convex domain in $\mathbf{C}^n$], Trans. Amer. Math. Soc. **289** (1985), no. 1, 345-353.

[Zhu1] K. Zhu. *BMO and Hankel operators on Bergman spaces*, Pacific J. Math. **155** (1992), 377-395.

[Zhu2] K. Zhu. *Compact composition operators on Bergman spaces of the unit ball*, Houston J. Math., to appear.

[ZhuStes] K. Zhu/M. Stessin. *Composition operators induced by symbols defined on a polydisk*, to appear in J. Math. Anal. Appl..