5.1 Antiderivatives and Indefinite Integration

Suppose that \( f(x) = 5x^4 \). Can we find a function \( F(x) \) whose derivative is \( f(x) \)?

**Definition.** A function \( F \) is an antiderivative of \( f \) on an interval \( I \) if \( F'(x) = f(x) \) for all \( x \) in \( I \).

**Note.** Observe that \( F_1(x) = x^5 \), \( F_2(x) = x^5 + 2 \), and \( F_3(x) = x^5 - \sqrt{7} \) are all antiderivatives of \( f \). To represent all possible antiderivatives of \( f(x) \), we write \( F(x) = x^5 + C \) where \( C \) is a constant.

**Theorem 5.1.** If \( F \) is an antiderivative of \( f \) on an interval \( I \), then \( G \) is an antiderivative of \( f \) on the interval \( I \) if and only if \( G \) is of the form \( G(x) = F(x) + C \), for all \( x \) in \( I \) where \( C \) is a constant.

**Note.** We call \( C \) the constant of integration and \( G \) the general antiderivative of \( f \).

**Notation.** From the last section, for

\[
\frac{dy}{dx} = f(x)
\]

we wrote

\[
dy = f(x) \, dx
\]

The process of finding all solutions to this equation is called indefinite integration or antidifferentiation. We write

\[
y =
\]
Note. There is an “inverse” relationship between differentiation and indefinite integration. That is

- \( \int F'(x)\, dx = \)

- If \( \int f(x)\, dx = F(x) + C \), then
  \[ \frac{d}{dx} \left[ \int f(x)\, dx \right] = \]

Basic Integration Rules

Compare these with the basic differentiation rules that we already know.

1. \( \int 0\, dx = \)
2. \( \int k\, dx = \)
3. \( \int kf(x)\, dx = \)
4. \( \int [f(x) \pm g(x)]\, dx = \)
5. \( \int x^n\, dx = \)
6. \( \int \frac{1}{x}\, dx = \)
7. \( \int \cos x\, dx = \)
8. \( \int \sin x\, dx = \)
9. \( \int \sec^2 x\, dx = \)
10. \( \int \csc^2 x\, dx = \)
11. \( \int \sec x\tan x\, dx = \)
12. \( \int \csc x\cot x\, dx = \)
Example 1. Evaluate $\int 5 \, dx$

Example 2. Evaluate $\int 5x \, dx$

Example 3. Evaluate $\int \frac{1}{x^4} \, dx$

Example 4. Evaluate $\int \sqrt[3]{x} \, dx$

Example 5. Evaluate $\int 2 \cos x \, dx$

Example 6. Evaluate $\int \frac{5}{x} \, dx$
Example 7. Evaluate \( \int dx \)  

Example 8. Evaluate \( \int (x + 4) \, dx \)

Example 9. Evaluate \( \int (x^2 + 5x + 4) \, dx \)

Example 10. Evaluate \( \int \frac{3x + 2}{\sqrt{x}} \, dx \)  

Example 11. Evaluate \( \int \frac{\sin x}{\cos^2 x} \, dx \)
Example 12. Near the surface of the earth, the acceleration of a falling body due to gravity is $-32$ feet per second per second, provided that air resistance is neglected. If an object is thrown upward from an initial height of 1000 feet with an initial velocity of 50 feet per second, find its velocity and height four seconds later.
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5.2 Area

Sigma Notation

The sum of $n$ terms $a_1, a_2, a_3, \ldots, a_n$ may be denoted by

$$\sum_{i=1}^{n} a_i =$$

where $i$ is the index of summation, $a_i$ is the $i^{th}$ term (or general term) of the sum, and 1 and $n$ are the lower and upper bounds of summation respectively.

Example 1. Expand $\sum_{i=1}^{5} i$.

Example 2. Expand $\sum_{i=3}^{6} i^2$.

Example 3. Expand $\sum_{k=1}^{n} \frac{1}{n}(k^2 - 1)$.

Example 4. Expand $\sum_{i=1}^{n} f(x_i) \Delta x$. 
Properties of Sigma Notation

1. $\sum_{i=1}^{n} ka_i =$

2. $\sum_{i=1}^{n} (a_i \pm b_i) =$

Theorem 5.2.

1. $\sum_{i=1}^{n} c =$

2. $\sum_{i=1}^{n} i =$

3. $\sum_{i=1}^{n} i^2 =$

4. $\sum_{i=1}^{n} i^3 =$

Example 5. Find the summation formula for $\sum_{i=1}^{n} \frac{i^2 + 1}{n^3}$. Use your result to evaluate the sum for $n = 10, 100, 1000, 10000$. 
Example 6. Use four rectangles to find two approximations of the area of the region lying between the graph of $f(x) = 2^x$ and the $x$-axis between the the vertical lines $x = 0$ and $x = 2$. 
To approximate the area under the curve of a nonnegative function bounded by the $x$-axis and the line $x = a$ and $x = b$...
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**Theorem 5.3.** Let $f$ be continuous and nonnegative on the interval $[a,b]$. The limits of both the lower and upper sums exist and

$$\lim_{n \to \infty} s(n) =$$

where $\Delta x = (b - a)/n$ and $f(m_i)$ and $f(M_i)$ are the minimum and maximum values of $f$ on the subinterval.

**Definition.** Let $f$ be continuous and nonnegative on the interval $[a,b]$. The area of the region bounded by the graph of $f$, the $x$-axis, and the vertical lines $x = a$ and $x = b$ is

area =

where $\Delta x = (b - a)/n$.

**Example 7.** Find the area of the region bounded by the graph of $f(x) = x^2$, the $x$-axis, and the vertical lines $x = 0$ and $x = 1$. 
5.3 The Riemann Sum and the Definite Integral

The following definition allows us to find the area under a function when the width of the rectangles are not all the same.

**Definition.** Let \( f \) be defined on the closed interval \([a, b]\), and let \( \Delta \) be a partition of \([a, b]\) given by \(a =\)

where \( \Delta x_i \) is the width of the \(i\)th subinterval. If \( c_i \) is any point in the \(i\)th subinterval, then the sum

\[
\sum_{i=1}^{n} f(c_i) \Delta x_i
\]

is called a **Riemann sum** of \( f \) for the partition \( \Delta \).

**Note.** This is a generalization of the upper and lower sum method we developed earlier.

We define the width of the largest subinterval of a partition \( \Delta \) to be the **norm** of the partition denoted by \( \| \Delta \| \).

If every subinterval is of equal width, then the partition is said to be **regular** and the norm is given by

\[
\| \Delta \| =
\]
5.3 The Riemann Sum and the Definite Integral

If we are using a general partition, then the norm of the general partition is related to the number of subintervals of \([a, b]\) by

\[
b - a \quad \|\Delta\|
\]

and as \(\|\Delta\| \to 0\), then this implies that \(n \to \infty\).

The converse of this statement is not true in general. It is only true in the case of a regular partition. For instance, consider the partition \(\Delta_n = 2^{-n}\) of the interval \([0, 1]\).

Thus,

\[
\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i) \Delta x_i = L
\]

exists provided that for each \(\varepsilon > 0\), there exists a \(\delta > 0\) with the property that for every partition with \(\|\Delta\| < \delta\) implies that

\[
\left| L - \sum_{i=1}^{n} f(c_i) \Delta x_i \right| < \varepsilon
\]

for any \(c_i\) in the \(i^{th}\) subinterval of \(\Delta\).

**Definition.** If \(f\) is defined on the closed interval \([a, b]\) and the limit

\[
\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i) \Delta x_i
\]

exists (as described above), then \(f\) is **integrable** on \([a, b]\) and the limit is denoted by

\[
\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i) \Delta x_i =
\]

and is called the **definite integral** of \(f\) from \(x = a\) to \(x = b\). The number \(a\) is the **lower limit of integration** and the number \(b\) is the **upper limit of integration**.
Theorem 5.4. If a function $f$ is continuous on the closed interval $[a, b]$, then $f$ is integrable on $[a, b]$.

Example 1. Evaluate $\int_{-1}^{2} 2x \, dx$
Theorem 5.5. If a function $f$ is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of $f$, the $x$-axis, and the vertical lines $x = a$ and $x = b$ is given by

area =

Example 2. Sketch the region corresponding to $\int_{1}^{5} 3 \, dx$ and evaluate the integral by using the geometry of the region.

Example 3. Sketch the region corresponding to $\int_{-3}^{3} \sqrt{9 - x^2} \, dx$ and evaluate the integral by using the geometry of the region.
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**Theorem 5.6.** If a function $f$ is integrable on the three closed intervals determined by $a < c < b$, then
\[
\int_a^b f(x) \, dx =
\]

Example 4. Evaluate $\int_{-4}^4 |x| \, dx$.

**Theorem 5.7.** Suppose that $k$ is a real number and $f$ and $g$ are integrable on $[a,b]$. Then, the function $kf$ and $f \pm g$ are integrable on $[a,b]$ and

1. $\int_a^b kf(x) \, dx =$
2. $\int_a^b [f(x) \pm g(x)] \, dx =$

Example 5. For the definite integral, we can write
\[
\int_1^3 (x^2 + 5x + 6) \, dx =
\]
Theorem 5.8.

1. If $f$ is integrable and nonnegative on the closed interval $[a, b]$, then

2. If $f$ and $g$ are integrable on the closed interval $[a, b]$ and $f(x) \leq g(x)$ for every $x$ in $[a, b]$, then

Properties of the Definite Integral

1. If $f$ is defined at $x = a$, then

2. If $f$ is integrable on $[a, b]$, then

Example 6. For the given definite integrals, we can write

$$\int_{\pi}^{\pi} e^x \, dx =$$

and

$$\int_{0}^{0} (x + 4) \, dx =$$
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5.4 The Fundamental Theorem of Calculus

Theorem 5.9 (Fundamental Theorem of Calculus Part I). If a function $f$ is continuous on the closed interval $[a, b]$ and $F$ is an antiderivative of $f$ on $[a, b]$, then

$$\int_{a}^{b} f(x) \, dx =$$

**Note.** We lose the constant of integration when performing definite integration.
Example 1. Evaluate $\int_1^2 (x^2 + 5x + 6) \, dx$.

Example 2. Evaluate $\int_0^4 \sqrt{x} \, dx$.

Example 3. Evaluate $\int_0^{\pi} \tan x \, dx$.

Example 4. Find the area of the region bounded by the graph of $y = \frac{1}{x}$, the $x$-axis, and the vertical lines $x = 1$ and $x = e^2$. 
Theorem 5.10 (The Mean Value Theorem for Integrals). If $f$ is continuous on the closed interval $[a, b]$, then there exists a number $c$ in the closed interval $[a, b]$ such that

$$\int_{a}^{b} f(x) \, dx =$$

Example 5. Use the Mean Value Theorem for Integrals to find all values of $c$ with $\int_{0}^{1} (3x^2 - 2x) \, dx$ that satisfy the theorem.
Definition. If $f$ is integrable on the closed interval $[a, b]$, then the \textit{average value} of $f$ on the interval is given by

average value $=$

Example 6. Find the average value of $f(x) = x^2 - x$ on the interval $[1, 4]$. 
5 Integration

**Accumulation Functions**

Define the *accumulation function*

\[ F(x) = \]

where \( F \) is a function of \( x \), \( f \) is a function of \( t \), and \( a \) is a constant.

**Example 7.** Evaluate \( F(x) = \int_0^x (1-t) \, dt \) for \( x = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1 \).
Theorem 5.11 (Fundamental Theorem of Calculus Part II). If $f$ is continuous on an open interval $I$ containing $a$, then, for every $x$ in the interval,

$$\frac{d}{dx} \left[ \int_a^x f(t) \, dt \right] =$$
Example 8. Find the derivative of \( \int_0^x \sqrt{1 - t^2} \, dt \).

Example 9. Find the derivative of \( \int_0^{x^3} \sin t \, dt \).
5.5 The $u$-Substitution

We want to now be able to work with integrands that have the form of the Chain Rule. Hence, recall that

**Theorem 5.12 (The $u$-Substitution).** Let $g$ be a function whose range is an interval $I$, and let $f$ be a function that is continuous on $I$. If $g$ is differentiable on its domain and $F$ is an antiderivative of $f$ on $I$, then

$$\int f(g(x))g'(x)\,dx =$$

If $u = g(x)$, then $du = g'(x)\,dx$ and

$$\int f(u)\,du =$$

**Example 1.** Evaluate $\int (x^2 + 4)(2x)\,dx$.

**Example 2.** Evaluate $\int 2e^{2x}\,dx$.

**Example 3.** Evaluate $\int x(x^2 + 4)^2\,dx$. 
**Example 4.** Evaluate $\int \sqrt{3x-1} \, dx$.

**Example 5.** Evaluate $\int x \sqrt{3x-1} \, dx$.

**Example 6.** Evaluate $\int \sin(5x) \cos(5x) \, dx$. 
Theorem 5.13 (The General Power Rule for Integration). If $g$ is a differentiable function of $x$, then
\[
\int [g(x)]^n g'(x) \, dx =
\]
If $u = g(x)$, then
\[
\int u^n \, du =
\]

Example 7. Evaluate $\int \frac{2x}{(1-x^2)^2} \, dx$.

Example 8. Evaluate $\int \sin^2 x \cos x \, dx$. 
5 Integration

Theorem 5.14 (Change of Variables for Definite Integrals). If the function \( u = g(x) \) has a continuous derivative on the closed interval \([a, b]\) and \( f \) is continuous on the range of \( g \), then

\[
\int_{a}^{b} f(g(x))g'(x) \, dx =
\]

Example 9. Evaluate \( \int_{0}^{1} x(x^2 - 1)^3 \, dx \).

Example 10. Evaluate \( \int_{3}^{8} \frac{x}{\sqrt{x + 1}} \, dx \).
Theorem 5.15. Let $f$ be integrable on the closed interval $[-a, a]$.

1. If $f$ is an even function, then
\[ \int_{-a}^{a} f(x) \, dx = \]
2. If $f$ is an odd function, then
\[ \int_{-a}^{a} f(x) \, dx = \]

Example 11. Evaluate $\int_{-10}^{10} (x^3 + x) \, dx$.

Example 12. Evaluate $\int_{-\sqrt{2}}^{\sqrt{2}} (4 - x^2) \, dx$. 
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5.6 Numerical Integration

Theorem 5.16 (The Trapezoid Rule). Let \( f \) be continuous on \([a, b]\). Then,
\[
\int_{a}^{b} f(x) \, dx \approx y
\]

Example 1. Use the Trapezoid Rule to approximate \( \int_{1}^{2} \frac{dx}{x} \) for \( n = 5 \).
**Theorem 5.17.** Let \( p(x) = Ax^2 + Bx + C \), then

\[
\int_a^b p(x) \, dx =
\]

**Theorem 5.18 (Simpson’s Rule).** Let \( f \) be continuous on \([a, b]\). If \( n \) is even, then

\[
\int_a^b f(x) \, dx \approx
\]

**Example 2.** Use Simpson’s Rule to approximate \( \int_1^2 \frac{dx}{x} \) for \( n = 10 \).
Theorem 5.19.

1. If $f$ has a continuous second derivative on $[a, b]$, then the error of estimating $\int_a^b f(x) \, dx$ by the Trapezoid Rule is

$$E \leq$$

2. If $f$ has a continuous fourth derivative on $[a, b]$, then the error of estimating $\int_a^b f(x) \, dx$ by Simpson’s Rule is

$$E \leq$$

Example 3. Determine a value of $n$ such that the Trapezoid Rule will approximate $\int_1^2 \frac{dx}{x}$ with an error that is less than 0.0001.
5.7 Integration and the Natural Logarithm

Using what we know about the derivative of the natural logarithm,

**Theorem 5.20.** Let $u$ be a differentiable function of $x$.

1. $\int \frac{dx}{x} =$
2. $\int \frac{du}{u} =$
3. $\int \frac{u'dx}{u} =$

**Example 1.** Evaluate $\int \frac{1}{3x} \, dx$.

**Example 2.** Evaluate $\int \frac{1}{5x - 2} \, dx$. 
Example 3. Find the area of the region bounded by the graph of 
\[ y = \frac{2x}{x^2 + 1}, \] 
the x-axis, and the vertical lines \( x = 0 \) and \( x = e^2 \).

Example 4. Evaluate \( \int \frac{x^2 + 5x + 1}{x^2 + 1} \, dx \).
Example 5. Solve the differential equation $\frac{dy}{dx} = \frac{e}{x \ln x}$.

Example 6. Evaluate $\int \frac{ex}{(x+1)^2} \, dx$. 
Example 7. Evaluate $\int \cot x \, dx$.

Example 8. Evaluate $\int \csc x \, dx$. 
5.8 Integrating Inverse Trigonometric Functions

Theorem 5.21. Let \( u \) be a differentiable function of \( x \), and let \( a > 0 \).

1. \( \int \frac{du}{\sqrt{a^2 - u^2}} = \)

2. \( \int \frac{du}{a^2 + u^2} = \)

3. \( \int \frac{du}{u\sqrt{u^2 - a^2}} = \)
Example 1. Evaluate \( \int \frac{dx}{\sqrt{49 - x^2}} \)

Example 2. Evaluate \( \int \frac{dx}{3 + e^{2x}} \)

Example 3. Evaluate \( \int \frac{dx}{x \sqrt{9x^2 - 25}} \)
Example 4. Evaluate $\int \frac{dx}{\sqrt{x^4 - 9}}$

Example 5. Evaluate $\int \frac{x + 1}{\sqrt{9 - x^2}} \, dx$
Example 6. Evaluate \( \int_{-2}^{2} \frac{dx}{x^2 + 4x + 13} \)

Example 7. Evaluate \( \int \frac{5}{\sqrt{5x - x^2}} \, dx \)