§7.3 VOLUME: The SHELL method

In this section we will study an alternative method, the Shell method, for finding the volume of a solid revolution.

Consider the cylindrical shell (or tube) of thickness \( w \) at right.

Volume of 1 representative shell
\[
= (\text{volume of cylinder}) - (\text{volume of hole})
\]
\[
= \pi \left( p + \frac{w}{2} \right)^2 h - \pi \left( p - \frac{w}{2} \right)^2
\]
\[
= 2\pi phw
\]
\[
= 2\pi \text{(average radius)(height)(thickness)}
\]

Now, consider a horizontal rectangle of width \( \Delta y \) revolving about a line parallel to the \( x \)-axis. This rectangle generates a representative shell whose volume is:
\[
\Delta V = 2\pi \left[ p(y)h(y) \right] \Delta y
\]

The volume of the solid of revolution is then approximated by \( n \) such shells of thickness \( \Delta y \).

Volume of solid \( \approx \sum_{i=1}^{n} 2\pi \left[ p(y_i)h(y_i) \right] \Delta y \)

This approximation becomes better and better as \( n \rightarrow \infty \) (\( \|\Delta y\| \rightarrow 0 \)). So:

Volume of solid = \( \lim_{\|\Delta y\| \rightarrow 0} 2\pi \left[ p(y)h(y) \right] \Delta y \)
\[
= 2\pi \int_c^d \left[ p(y)h(y) \right] dy
\]

where \( p(y) = \text{distance from the shell to the axis of revolution} \), and \( h(y) = \text{height of the rectangle} \).
The Shell Method

To find the volume of a solid of revolution with the **shell method**, use one of the following, as shown in Figure 7.29.

<table>
<thead>
<tr>
<th>Horizontal Axis of Revolution</th>
<th>Vertical Axis of Revolution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volume = ( V = 2\pi \int_c^d p(y)h(y) , dy )</td>
<td>Volume = ( V = 2\pi \int_a^b p(x)h(x) , dx )</td>
</tr>
</tbody>
</table>

**Note:** In the Shell method, the representative rectangle is **always** parallel to the axis of revolution.

**Example 1:** Use the Shell method to find the volume of the solid formed by revolving about the y-axis the region bounded by \( y = x^2 + 4 \), \( y = 8 \), and \( x = 0 \).
Example 2: Find the volume of the solid formed by revolving the region bounded by 

\[ y = \frac{x}{2}, \quad y = 0 \quad \text{and} \quad x = 4 \]

about the line \( y = -2 \) (use the Shell method).

**Solution:** First, we solve for \( x \) in terms of \( y \) in the equation 

\[ y = \frac{x}{2} \]

(because the axis of revolution is horizontal, and thus, in the Shell method, the rectangle must be drawn horizontally also).

**Question:** Set up the integral to find the volume by using the Disk method.

- Sometimes, solving for \( x \) is very difficult (or even impossible). In such cases:
  - We must use a **vertical** rectangle (of width \( \Delta x \)), thus making \( x \) the variable of integration.
  - The position of the vertical rectangle in relation to the axis of revolution (i.e., perpendicular or parallel) determines the method to be used.

Read Example 5 in text (pg. 471).
COMPARISON of Disk and Shell methods

Disk method: Representative rectangle is perpendicular to the axis of revolution.

Shell method: Representative rectangle is parallel to the axis of revolution.

For any given volume problem, one method is more convenient to use than the other as in:
- Only 1 integral is required to find the volume (versus when 2 integrals are needed) – Read Example 3 in text (pg. 470); or,
- The integral is simple and “solvable” (versus a really long, complicated integral) – Read Example 4 in text (pg. 471).
Example 3: Find the volume of the solid formed by revolving the region bounded by
\[ y = e^{\frac{y}{2}}, \quad y = 0, \quad x = 0, \text{ and } x = 2 \]

about the x-axis.

Let’s see which of the 2 methods is better!!!

By the Shell method:

By the Disk method: