§8.4 TRIGONOMETRIC SUBSTITUTION

Objective: To study techniques for evaluating integrals involving the radicals $\sqrt{a^2 - u^2}$, $\sqrt{a^2 + u^2}$, and $\sqrt{u^2 - a^2}$.

General Approach: To eliminate the radical in the integrand, by using the following Pythagorean identities:

\[
\begin{align*}
\cos^2 \theta &= 1 - \sin^2 \theta, \\
\sec^2 \theta &= 1 + \tan^2 \theta, \\
\tan^2 \theta &= \sec^2 \theta - 1.
\end{align*}
\]

Note 1: The restrictions on $\theta$ ensure that the function that defines the substitution is one-on-one. [In fact, these are the same intervals over which the arcsine, arctangent, and arcsecant are defined.]

Note 2: Before applying trigonometric substitution, generally it’s always a good idea to check if any of the basic integration rules applies or not. [For purpose of practice, all integrals in these notes (section 7.4) require trig substitution!]

Trigonometric Substitution ($a > 0$)

1. For integrals involving $\sqrt{a^2 - u^2}$, let

\[ u = a \sin \theta. \]

Then $\sqrt{a^2 - u^2} = a \cos \theta$, where $-\pi/2 \leq \theta \leq \pi/2$.

2. For integrals involving $\sqrt{a^2 + u^2}$, let

\[ u = a \tan \theta. \]

Then $\sqrt{a^2 + u^2} = a \sec \theta$, where $-\pi/2 < \theta < \pi/2$.

3. For integrals involving $\sqrt{u^2 - a^2}$, let

\[ u = a \sec \theta. \]

Then $\sqrt{u^2 - a^2} = \pm a \tan \theta$, where $0 \leq \theta < \pi/2$ or $\pi/2 < \theta < \pi$.

Use the positive value if $u > a$ and the negative value if $u < -a$. 
Example 1: \[
\int \frac{1}{m \sqrt{9 - m^2}} \, dm
\]
Example 2: Find \[ \int \frac{dx}{\sqrt{4x^2 - 25}} \]
Example 3: Find \[ \int \frac{dx}{x^2 \sqrt{x^2 + 4}} \]

Trig substitution can be used to evaluate the 3 following integrals in Theorem 7.2. We will encounter these integrals several times in the remainder of the text.

**THEOREM 8.2 Special Integration Formulas \((a > 0)\)**

1. \[ \int \sqrt{a^2 - u^2} \, du = \frac{1}{2} \left( a^2 \arcsin \frac{u}{a} + u \sqrt{a^2 - u^2} \right) + C \]
2. \[ \int \sqrt{u^2 - a^2} \, du = \frac{1}{2} \left( u \sqrt{u^2 - a^2} - a^2 \ln |u + \sqrt{u^2 - a^2}| \right) + C, \quad u > a \]
3. \[ \int \sqrt{u^2 + a^2} \, du = \frac{1}{2} \left( u \sqrt{u^2 + a^2} + a^2 \ln |u + \sqrt{u^2 + a^2}| \right) + C \]
DEFINITE INTEGRALS: When using trig substitution to evaluate definite integrals, be very careful to check that the values of $\theta$ lie in the intervals discussed in the Procedure (see the Box, page 1 of 6).

Example 4: Evaluate \( \int_{\sqrt{3}}^{2} \frac{\sqrt{x^2 - 3}}{x} \, dx \)
Example 5: Evaluate \( \int_{-2}^{5} \frac{\sqrt{x^2 - 3}}{x} \, dx \)