§9.3 The INTEGRAL TEST; $p$-Series

In this and the following section, you will study several convergence tests that apply to series with positive terms.

**Theorem 9.10 The Integral Test**

If $f$ is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then

$$
\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_{1}^{\infty} f(x) \, dx
$$

either both converge or both diverge.

**Note 1:** When using the Integral Test, it’s NOT necessary to start at $n = 1$. For instance, when testing the series $\sum_{n=4}^{\infty} \frac{1}{(n-3)^2}$, we’d use the improper integral $\int_{4}^{\infty} \frac{1}{(x-3)^2} \, dx$.

**Example 1:** Discuss the convergence / divergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$

**Solution:**
**Note 2:** The series doesn’t always converge to the same value as that of the improper integral. Note that the Integral Test doesn’t state the limit value to which the series converges!!!

**Note 3:** It’s also NOT necessary that function $f'$ be always decreasing. **What’s important is that $f$ be ULTIMATELY decreasing** [i.e., $f$ is decreasing for $x \geq N$ for some constant $N$].

**Example 2:** Test the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence / divergence.

**Solution:**
**p-Series**

**Def:** A series of the form \( \sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots \) is called a *p-series*, where \( p \) is a positive constant.

For \( p = 1 \), the series \( \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots \) is called the **harmonic series**.

A general harmonic series is of the form \( \sum \frac{1}{an+b} \).

The *p*-series has a simple arithmetic test for convergence or divergence.

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**THEOREM 9.11  Convergence of *p*-Series**

The *p*-series

\[
\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots
\]

1. converges if \( p > 1 \), and
2. diverges if \( 0 < p \leq 1 \).

So, by the definition of the harmonic series, it’s divergent.

**Example 3:** Discuss the convergence / divergence of each series.

a) \( \sum_{n=1}^{\infty} \frac{2}{n^3} \)

b) \( 1 + \frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{9}} + \frac{1}{\sqrt[3]{16}} + \ldots \)
§9.4 COMPARISONS of SERIES

The two tests you study in this section allow you to compare one series having complicated terms with a simpler series whose convergence or divergence is known (or can be determined quite easily!!!).

Note 1: Remember that both parts of the Direct Comparison Test require $0 < a_n \leq b_n$ for all positive terms. Informally, this test says the following about the 2 series with positive terms:

1. If the “larger” series converges, the “smaller” series must also converge.
2. If the “smaller” series diverges, the “larger” series must also diverge.

Note 2: When choosing a series for comparison, you may disregard all but the dominant term / the Highest Powers of $n$ in both the numerator and denominator.

Example 1: Discuss the convergence / divergence of the series $\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n}$

Solution:
Example 2: Determine the convergence / divergence of \( \sum_{n=1}^{\infty} \frac{1}{2 + \sqrt{n}} \)

Solution:

The given series resembles \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \) (which is a divergent \( p \)-series with \( p = \frac{1}{2} < 1 \)).

But \( \frac{1}{2 + \sqrt{n}} < \frac{1}{\sqrt{n}} \) for all \( n \geq 1 \), does not meet the requirement of divergence (part 2 of Direct Comparison Test). So, try a different series: the harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \) (which is also divergent!).

And, term-by-term comparison yields \( \frac{1}{n} \leq \frac{1}{2 + \sqrt{n}} \) for all \( n > 4 \). (Do you see?)

\( \therefore \) By the Direct Comparison Test part 2, \( \sum_{n=1}^{\infty} \frac{1}{2 + \sqrt{n}} \) diverges.

Also read Examples 1 & 2 in textbook, pg. 625.

Now, a given series may closely resemble a \( p \)-series or a geometric series, yet you cannot establish the term-by-term comparison required to apply the Direct Comparison Test.

\( \rightarrow \) Under these circumstances, you may be able to apply a second comparison test called the Limit Comparison Test.

**THEOREM 9.13 Limit Comparison Test**

Suppose that \( a_n > 0, b_n > 0 \), and

\[ \lim_{n \to \infty} \left( \frac{a_n}{b_n} \right) = L \]

where \( L \) is finite and positive. Then the two series \( \sum a_n \) and \( \sum b_n \) either both converge or both diverge.

Note 3: Remember that the limit value \( L \) must be finite and positive in order for you to apply the Limit Comparison Test and have the correct conclusion. If \( L = 0 \), then you need to change the comparison series \( \{b_n\} \).
Example 3: Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence or divergence.

Solution:

Example 4: Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{n \cdot 2^n}{4n^3 + 1}$.

Solution:

A reasonable comparison would be with the series $\sum_{n=1}^{\infty} \frac{n \cdot 2^n}{n^3}$, or $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$. Then let’s find out whether the comparison series $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ is convergent or divergent. It is neither a geometric series nor a $p$-series, but:

$$\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{2^n \cdot \ln^2 n}{n^2} \neq 0$$

so by the $n$th-Term Test for Divergence, the series $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ is divergent. Now:

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \left[ \frac{n \cdot 2^n \cdot n^2}{4n^3 + 1} \right] = \lim_{n \to \infty} \left[ \frac{n^3}{4n^3 + 1} \right] = \frac{1}{4} > 0$$

∴ By the Limit Comparison Test, the given series $\sum_{n=1}^{\infty} \frac{n \cdot 2^n}{4n^3 + 1}$ diverges.

Also read Examples 3 & 4 in textbook, pgs. 626-627.