§9.8 POWER SERIES

Objective: To learn that several important types of functions \( f \) can be represented exactly by an infinite series called a power series.

**Definition of Power Series**

If \( x \) is a variable, then an infinite series of the form

\[
\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots
\]

is called a power series. More generally, an infinite series of the form

\[
\sum_{n=0}^{\infty} a_n (x - c)^n = a_0 + a_1 (x - c) + a_2 (x - c)^2 + \cdots + a_n (x - c)^n + \cdots
\]

is called a power series centered at \( c \), where \( c \) is a constant.

**Example 1:**

**Radius and Interval of Convergence**

A power series in \( x \) can be viewed as a function of \( x \):

\[
f(x) = \sum_{n=0}^{\infty} a_n (x - c)^n
\]

where the “domain” of \( f \) is the set of all \( x \) for which the power series converges.

The following important theorem states that the domain of a power series can take 3 basic forms: (1) a single point, (2) an interval centered at \( c \), or (3) the entire real line.
Example 2: Find the Radius of Convergence.

a) \[ \sum_{n=1}^{\infty} \frac{(x-3)^n}{n} \]
b) \[ \sum_{n=0}^{\infty} (n!)x^n \]

c) \[ \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \]

**INTERVAL OF CONVERGENCE**

Note that for a power series whose radius of convergence is a finite number \( R \), the previous theorem says *nothing about the convergence at the endpoints* of the interval of convergence.

So, **each endpoint must be tested separately for convergence or divergence**.
Example 3: Find the Interval of Convergence.

a) \[ \sum_{n=0}^{\infty} \frac{(-1)^n(x+1)^n}{2^n} \]

b) \[ \sum_{n=1}^{\infty} \frac{x^n}{n^2} \]
Once a power series is defined as a function, it’s natural for us to wonder how we can determine the characteristics of the function. Is it still continuous? Differentiable? The next theorem answers these questions.

**THEOREM 9.21 Properties of Functions Defined by Power Series**

If the function given by

\[ f(x) = \sum_{n=0}^{\infty} a_n (x - c)^n \]

\[ = a_0 + a_1(x - c) + a_2(x - c)^2 + a_3(x - c)^3 + \cdots \]

has a radius of convergence of \( R > 0 \), then, on the interval \((c - R, c + R)\), \( f \) is differentiable (and therefore continuous). Moreover, the derivative and antiderivative of \( f \) are as follows.

1. \( f'(x) = \sum_{n=1}^{\infty} n a_n (x - c)^{n-1} \)

\[ = a_1 + 2a_2(x - c) + 3a_3(x - c)^2 + \cdots \]

2. \[ \int f(x) \, dx = C + \sum_{n=0}^{\infty} a_n \frac{(x - c)^{n+1}}{n + 1} \]

\[ = C + a_0(x - c) + a_1 \frac{(x - c)^2}{2} + a_2 \frac{(x - c)^3}{3} + \cdots \]

The radius of convergence of the series obtained by differentiating or integrating a power series is the same as that of the original power series. The interval of convergence, however, may differ as a result of the behavior at the endpoints.
Example 4: Consider the function \( f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots \)

Find the Interval of Convergence for each of the following:

a) \( \int f(x) \, dx \)  
b) \( f(x) \)  
c) \( f'(x) \)

Solution: [pg. 665 in text]