CONTENT This exam will cover the material discussed in sections 4.1–4.5, and 4.7.

TOPICS You should be comfortable with the following topics:

Absolute extrema on a closed interval, relative extrema, critical numbers, Rolle's Theorem, the Mean Value Theorem, increasing and decreasing functions, the first derivative test, concavity, points of inflection, the second derivative test, L'Hôpital's rule, indeterminate forms, optimization problems.

FORMULAS You should have the following formulas memorized.

The Mean Value Theorem

If f is continuous on $[a,b]$ and differentiable on (a,b), then there exists a $c \in (a,b)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

PRACTICE PROBLEMS

1. (4.1) Consider the function, $f(\theta) = \cos^2\theta + \sin\theta$ (without using the graphing utility on your calculator).

- **(a)** Find all critical values in $\left[-\frac{\pi}{3}, \frac{2\pi}{3}\right]$.
- **(b)** Find where the absolute extrema occur.
- **(c)** Find the values of the absolute extrema.

2. (4.2) For each function on the given closed interval, explain why Rolle's Theorem does not apply.

- **(a)** $f(x) = \dfrac{x}{x-1}, \quad [2,3]$
- **(b)** $g(x) = \tan x, \quad [0,\pi]$
- **(c)** $h(x) = |x-2|, \quad [-2,6]$

3. (4.2) Consider the function on the closed interval $[4,6]$.

$$f(x) = x^3 - 6x^2 - 4x + 30$$

- **(a)** The Mean Value Theorem guarantees the existence of a point $c \in [4,6]$, such that $f'(c)$ is equal to what value?
- **(b)** Find the point(s) c.

4. (4.3) Consider the function

$$g(x) = 4\ln(x) - \frac{x^3}{6}$$

- **(a)** What is the domain of g?
- **(b)** Find the open interval(s) on which g is increasing and decreasing.
- **(c)** Find all relative extrema.

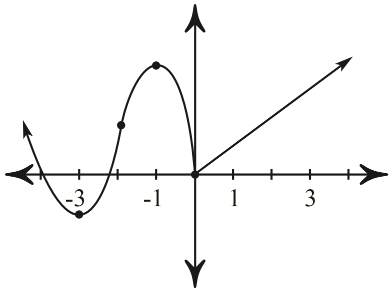
5. (4.3) Find the points of inflection and discuss the concavity of the graph of the function.

$$f(x) = x(x-3)^3$$

6. (4.3) Use the Second Derivative Test, if possible, to find all relative extrema.

- **(a)** $g(x) = \frac{1}{3}x^3 - x^2 - 15x + 1$
- **(b)** $y = xe^{-x}$
- **(c)** $y = x^3 - 9x^2 + 27x$

7. (4.3) The graph of f is shown.



a) Where is $f'(x)$ positive and where is it negative?

b) List the x-values of all critical points of f.

d) Use part (a) and (b) to sketch a graph of what $f'(x)$ might look like.

8. (4.4) Use L'Hôpital's rule to evaluate each limit.

(a) $\lim_{x \to 0} \frac{\sin^2 x}{5x^2}$

(b) $\lim_{x \to \infty} \frac{\ln \sqrt{x}}{x^4}$

(c) $\lim_{x \to 1} \frac{e^{2x} - e^2}{\ln x}$

9. (4.5) Consider the curve:

$$f(x) = \frac{x^2 \ln(x^2+1) - \ln(x^2+1)}{x^2 - 1}$$

(a) What is the domain?

(b) Find all x- and y-intercepts

(c) Discuss the symmetry.

(d) Identify any holes in the graph or asymptotes.

(e) On what intervals is f increasing/decreasing?

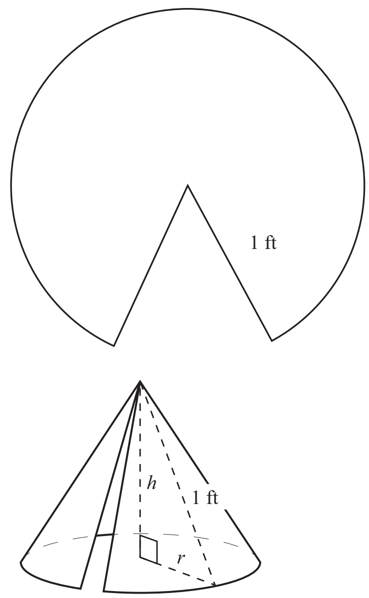
(f) Determine all relative extrema.

(g) Discuss the concavity of f.

(h) Identify any inflection points.

10. (4.7) A cylindrical can is to hold 648π cm^3 of liquid. If the material for the sides and the top costs \$0.15/cm^2 and the material for the bottom costs \$0.30/cm^2, find the height and radius that will minimize the cost of manufacture the can.

11. (4.7) A cone is made from a circular sheet of radius 1 foot by cutting out a sector and gluing the cut edges of the remaining piece together. What is the maximum volume attainable for the cone? The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$



Answers

1. (a) $(\frac{\pi}{2}, 1), (\frac{\pi}{6}, \frac{5}{4})$

(b) absolute maximum at $x = \frac{\pi}{6}$; absolute minimum at $x = -\frac{\pi}{3}$

(c) absolute maximum is $\frac{5}{4}$; absolute minimum is $\frac{1-2\sqrt{3}}{4}$

2. (a) $f(2) \neq f(3)$

(b) g is not continuous on $[\,0, \pi\,]$

(c) h is not differentiable on $(\,-2, 6\,)$

3. (a) 12

(b) $c = \dfrac{6 + 2\sqrt{21}}{3}$

4. (a) $(0, \infty)$

(b) increasing on $(0, 2)$; decreasing on $(2, \infty)$

(c) relative maximum at $(2, (4\ln 2 - \frac{4}{3}))$

5. Inflection points $(\frac{3}{2}, -\frac{81}{16})$ and $(3, 0)$; concave up on $(-\infty, \frac{3}{2}) \cup (3, \infty)$, concave down on $(\frac{3}{2}, 3)$

6. (a) relative maximum at $x = -3$, relative minimum at $x = 5$.

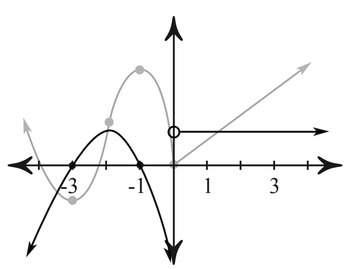
(b) relative maximum at $x = 1$

(c) no relative extrema

7. (a) positive on $(-3, -1) \cup (0, \infty)$; negative on $(-\infty, -3) \cup (-1, 0)$

(b) $x = -3, -1$, and 0

(c)



8. (a) $\dfrac{1}{5}$

(b) 0

(c) $2e^2$

9. (a) $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

(b) x- and y-int: $(0, 0)$

(c) symmetric w.r.t. the y-axis

(d) holes at $(-1, \ln(2))$ and $(1, \ln(2))$, no VA, no HA

(e) decreasing on $(-\infty, -1) \cup (1, 0)$, increasing on $(0, 1) \cup (1, \infty)$

(f) relative minimum at $(0, 0)$

(g) concave up on $(-1, 1)$, concave down on $(-\infty, -1) \cup (1, \infty)$

(h) none (note: $x = \pm 1$ is not on the curve)

10. radius = 6 cm, height = 18 cm

11. $\dfrac{2\pi\sqrt{3}}{27}$ ft^3