The Differential Operator

We define the linear operator D by $D(f) = f'$, called the **derivative operator**. Higher-order derivative operators can be defined by a composition, $D^k = D(D^{k-1})$ for $k = 1, 2, 3, ...$ and

$$D^k(f) = \frac{d^k}{dx^k}[f(x)]$$

For example, $D(\cos x) = -\sin x$, and $D^3(x^4) = D^2(4x^3) = D(12x^2) = 24x$.

Differential equations can be written using the derivative operator. The equation $y'' + 3y' - 21y = 0$, can be written $D^2(y) + 3D(y) - 21y = 0$ or simply, $(D^2 + 3D - 21)y = 0$. In this example, we call the combination of derivative operators applied to y, a **linear differential operator**. In general, the linear differential operator of order n is

$$L = D^n + a_1 D^{n-1} + a_2 D^{n-2} + ... + a_{n-1} D + a_n$$

Example 1 Apply the differential operator $L = D^2 + 4xD$ to the function, $y = x^3 + \cos x$.

$$\begin{aligned} Ly &= (D^2 + 4xD)(x^3 + \cos x) \\ &= D^2(x^3 + \cos x) + 4xD(x^3 + \cos x) \\ &= D(3x^2 - \sin x) + 4x(3x^2 - \sin x) \\ &= 6x - \cos x + 12x^3 - 4x \sin x \end{aligned} \tag{1}$$

Example 2 Apply the differential operator $L = (D - 3)^2$ to the function, $y = xe^{3x}$.

$$\begin{aligned} Ly &= (D-3)^2 xe^{3x} \\ &= (D-3)(D(xe^{3x}) - 3xe^{3x}) \\ &= (D-3)(e^{3x} + 3xe^{3x} - 3xe^{3x}) \\ &= (D-3)e^{3x} \\ &= 3e^{3x} - 3e^{3x} \\ &= 0 \end{aligned} \tag{2}$$

In this example, notice $Ly = 0$. We say the differential operator L "annihilated" y.

The Annihilator Method

The annihilator method can be used to transform the non-homogeneous linear equation of the form

$$y'' + p(x)y' + q(x)y = f(x)$$

into a homogeneous equation by multiplying both sides by a linear differential operator $A(D)$, that will "annihilate" the term $f(x)$.

Example 3 Use the annihilator method to convert the non-homogeneous equation to a homogeneous equation.

$$y'' - 12y' + 32y = 4\cos 3x$$

The term $4\cos 3x$ will be annihilated by $A(D) = D^2 + 9$. First rewrite the equation using the derivative operator, then apply $A(D)$ to both sides.

$$\begin{aligned}
(D^2 - 12D + 32)y &= 4\cos 3x \\
\implies (D^2 + 9)(D^2 - 12D + 32)y &= (D^2 + 9)(4\cos 3x) \\
\implies (D^2 + 9)(D^2 - 12D + 32)y &= 0 \\
\implies (D^2 + 9)(D - 4)(D - 8)y &= 0
\end{aligned}$$

We can now proceed to solve this differential equation using the method of undetermined coefficients, with $y_c = c_1e^{4x} + c_2e^{8x}$ and the trial solution for y_p, $y_p = A\cos 3x + B\sin 3x$.

Determining the differential operator that annihilate a given $f(x)$ should be done after sufficient practice solving homogeneous linear equations. However, the following table is a summary of common annihilators.

$\boldsymbol{f}(x)$	**Annihilator**
$a_0 + a_1x + a_2x^2 + ... + a_nx^n$	D^{n+1}
e^{rx}	$D - r$
x^ne^{rx}	$(D - r)^{n+1}$
$\cos bx$ or $\sin bx$	$D^2 + b^2$
$x^n\cos bx$ or $x^n\sin bx$	$(D^2 + b^2)^{n+1}$
$e^{ax}\cos bx$ or $e^{ax}\sin bx$	$(D - a)^2 + b^2 = D^2 - 2aD + a^2 + b^2$
$x^ne^{ax}\cos bx$ or $x^ne^{ax}\sin bx$	$((D - a)^2 + b^2)^{n+1} = (D^2 - 2aD + a^2 + b^2)^{n+1}$

Note: To annihilate the sum of two functions $f(x) + g(x)$, if $A_1(D)$ annihilates $f(x)$ and $A_2(D)$ annihilates $g(x)$, then use the composition of the two annihilators, $(A_1(D))(A_2(D))$.

For problems 1–2, apply the given differential operator, L to the function $f(x)$.

1. $L = D^2 + 5x^2D - 2x$, $f(x) = x^2 - \ln(x)$

2. $L = D^2 + 9$, $f(x) = 13\sin 3x$

3. Verify both differential operators, $L_1 = (D-2)D^2$ and $L_2 = D^2(D-2)$ will annihilate $f(x) = e^{2x} - 3x$, hence showing they are commutative.

For problems 4–7, determine the annihilator of the given function, then verify your answer by applying it to the function.

4. $f(x) = 2e^{-6x}$

5. $f(x) = 17\cos 5x$

6. $f(x) = xe^{4x}$

7. $f(x) = e^{-x} - 10x + 5$

For problems 8–12, determine the annihilator of the given function.

8. $f(x) = e^{-2x}(2\cos x + 8\sin x)$

9. $f(x) = (7 + x^2)e^{2x}$

10. $f(x) = e^{3x}(x - 8\sin 7x) + 3x^2\cos x$

11. $f(x) = 12x^3e^{-x}(\cos 5x)$

12. $f(x) = \sin^2 x$

For problems 13–15, apply the annihilator method and write the corresponding homogeneous equation using differential operator notation, then determine an appropriate trial solution. Do not solve for the constants.

13. $y'' + 13y' - 30y = e^{3x}$

14. $y'' + 8y' + 16y = 2xe^{-4x}$

15. $y'' + 4y' + 13y = 9e^{-2x}\sin 3x$

Answers

1. $8x^3 - 5x + 2 + \dfrac{1}{x^2} + 2x \ln x$

2. 0

3. $(D-2)D^2(e^{2x} - 3x) = 0$; $D^2(D-2)(e^{2x} - 3x) = 0$

4. $D + 6$

5. $D^2 + 25$

6. $(D-4)^2$

7. $(D+1)D^2$

8. $D^2 + 4D + 5$

9. $(D-2)^3$

10. $(D-3)^2(D^2 - 6D + 58)(D^2+1)^3$

11. $(D^2 + 2D + 26)^4$

12. $D(D^2+4)$

13. $(D-3)(D+15)(D-2)y = 0$; $y_p = Ae^{3x}$

14. $(D+4)^4 y = 0$; $y_p = Ax^2e^{-4x} + Bx^3e^{-4x}$

15. $(D^2 + 4D + 13)^2 y = 0$; $y_p = xe^{-2x}(A\cos 3x + B \sin 3x)$