First-Order Linear DE

Form: $y'(x) + p(x)y(x) = q(x)$

Integrating factor: $I(x) = e^{\int p(x)\,dx}$

Bernoulli Equations

Form: $y' + p(x)y = q(x)y^n$

Strategy: Divide through by y^n, multiply by $1-n$ then use the integrating factor:

$$I(x) = e^{\int (1-n)p(x)\,dx}$$

Homogeneous DE

Form: $\frac{dy}{dx} = f(x, y)$, where $f(tx, ty) = f(x, y)$

Strategy: Let $y = ux$, DE becomes separable

Exact DE

The DE $F_x(x, y)\,dx + F_y(x, y)\,dy = 0$ is exact if $F_{xy}(x, y) = F_{yx}(x, y)$.

The solution is $F(x, y) = C$

Integrating Factor

If $M(x, y)\,dx + N(x, y)\,dy = 0$ is not exact:

a) If $p(x) = \frac{M_y - N_x}{N}$ is independent of y, use the integrating factor: $\mu(x) = e^{\int p(x)\,dx}$

b) If $q(y) = \frac{N_x - M_y}{M}$ is independent of x, use the integrating factor: $\mu(y) = e^{\int q(y)\,dy}$

Euler's Method

Given $y' = f(x, y) \quad y(x_0) = y_0$

$$y_{i+1} = y_i + hf(x_i, y_i)$$

1st and 2nd-Order Circuits

$\frac{di}{dt} + \frac{R}{L}i + \frac{1}{LC}q = \frac{1}{L}E(t)$

where $i(t) = \frac{dq}{dt}$

Mixing Problems

$W'(t) = R_1 - R_2$

$Q'(t) = C_1R_1 - \frac{Q}{W(t)}R_2$

Variation of Parameters

Form: $y'' + a(x)y' + b(x)y = F(x)$, with solutions to the homogeneous case: y_1 and y_2

$y_p = y_1 \int \frac{-y_2F(x)}{W(y_1, y_2)}\,dx \; + \; y_2 \int \frac{y_1F(x)}{W(y_1, y_2)}\,dx$

Spring Problems

$$my''(t) + cy'(t) + ky(t) = F(t)$$

where m is the mass of the weight, c is the damping constant, k is the spring constant, and F is an external force.

Cauchy-Euler Equations

Form: $ax^2y'' + bxy' + cy = 0$

Let $x = e^u$ which leads to

$$ar(r-1) + br + c = 0$$

and $y(u) = e^{ru}$, substitute back to find $y(x)$

Method of Frobenius For $x^2y'' + xp(x)y' + q(x)y = 0$, r_1 and r_2 are the real roots of the indicial equation: $r(r-1) + p_0r + q_0 = 0$

1. If $r_1 - r_2$ is not an integer,

$$y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1} \quad \text{and} \quad y_2 = \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

2. If $r_1 - r_2$ is a positive integer,

$$y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1} \quad \text{and} \quad y_2 = Ay_1 \ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

where A is a constant and may turn out to be zero.

3. If $r_1 = r_2 = r$,

$$y_1 = \sum_{n=0}^{\infty} a_n x^{n+r} \quad \text{and} \quad y_2 = y_1 \ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r}$$

Common Laplace Transforms

$\mathcal{L}[t^n] = \dfrac{n!}{s^{n+1}}$

$\mathcal{L}[e^{at}] = \dfrac{1}{s-a}$

$\mathcal{L}[\sin bt] = \dfrac{b}{s^2+b^2}$

$\mathcal{L}[\cos bt] = \dfrac{s}{s^2+b^2}$

$\mathcal{L}[\sinh bt] = \dfrac{b}{s^2-b^2}$

$\mathcal{L}[\cosh bt] = \dfrac{s}{s^2-b^2}$

$\mathcal{L}[\delta(t-a)] = e^{-as}$

Laplace Transform of Derivatives

$$\mathcal{L}[y^{(n)}] = s^n\mathcal{L}[y] - s^{n-1}y(0) - s^{n-2}y'(0) - \ldots$$
$$\ldots - sy^{(n-2)}(0) - y^{(n-1)}(0)$$

The First Shifting Theorem

If $\mathcal{L}[f(t)] = F(s)$ then

$$\mathcal{L}[e^{at}f(t)] = F(s-a)$$

The Second Shifting Theorem

If $\mathcal{L}[f(t)] = F(s)$ then

$$\mathcal{L}[u(t-a)f(t-a)] = e^{-as}F(s)$$

Convolution Product

$$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$$