Consider the equation

$$x^2y'' + xp(x)y' + q(x)y = 0 \tag{1}$$

where $x = 0$ is a **regular singular point**. Since $p(x)$ and $q(x)$ are analytic at $x = 0$, they can be represented as

$$p(x) = \sum_{n=0}^{\infty} p_n x^n \quad \text{and} \quad q(x) = \sum_{n=0}^{\infty} q_n x^n, \quad \text{for } |x| < R$$

for some real $R > 0$. Let r_1 and r_2 be real roots to the **indicial equation**:

$$r(r-1) + p_0 r + q_0 = 0 \tag{2}$$

with $r_1 \geq r_2$.

Theorem (Method of Frobenius)

The two linear independent solutions to (1) on $(0, \infty)$, with a regular singular point at $x = 0$, and roots r_1 and r_2 to the indicial equation (2) can be determined as follows:

1. If $r_1 - r_2$ is not an integer,

$$y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1} \quad \text{and} \quad y_2 = \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

2. If $r_1 - r_2$ is a positive integer,

$$y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1} \quad \text{and} \quad y_2 = Ay_1 \ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

where A is a constant and may turn out to be zero.

3. If $r_1 = r_2 = r$,

$$y_1 = \sum_{n=0}^{\infty} a_n x^{n+r} \quad \text{and} \quad y_2 = y_1 \ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r}$$

Exercises

For each equation, (a) find the roots of the indicial equation and give the general form of the solution, (b) determine the recurrence relation for a_n, then (c) find two linearly independent solutions on $(0, \infty)$. If you cannot determine a general pattern for the coefficients of your power series, write the first 4 non-zero terms.

1. $3x^2y'' - x(x+8)y' + 6y = 0$

2. $4x^2y'' + 3xy' + xy = 0$

3. $x^2y'' + x^2y' - 2y = 0$

4. $x^2y'' + xy' - (1+x)y = 0$

5. $x^2y'' + 5xy' + (x+4)y = 0$

6. $x^2y'' - x(x+3)y' + 4y = 0$

Answers

1. (a) $r_1 = 3$, $r_2 = \frac{2}{3}$; $y_1 = \sum_{n=0}^{\infty} a_n x^{n+3}$; $y_2 = \sum_{n=0}^{\infty} b_n x^{n+2/3}$;

(b) $a_n = \dfrac{n+2}{n(3n+7)} a_{n-1}$; $b_n = \dfrac{3n-1}{3n(3n-7)} b_{n-1}$

(c) $y_1 = a_0 x^3 \left[1 + \sum_{n=1}^{\infty} \dfrac{(n+1)(n+2)}{2\prod_{k=1}^{n}(3k+7)} x^n\right]$; $y_2 = b_0 x^{2/3} \sum_{n=0}^{\infty} \dfrac{(3n-4)(3n-1)}{4 \cdot n! \cdot 3^n} x^n$

2. (a) $r_1 = \frac{1}{4}$, $r_2 = 0$; $y_1 = \sum_{n=0}^{\infty} a_n x^{n+1/4}$; $y_2 = \sum_{n=0}^{\infty} b_n x^n$;

(b) $a_n = -\dfrac{1}{n(4n+1)} a_{n-1}$; $b_n = -\dfrac{1}{n(4n-1)} b_{n-1}$

(c) $y_1 = a_0 x^{1/4} \left[1 + \sum_{n=1}^{\infty} \dfrac{(-1)^n}{n!\prod_{k=1}^{n}(4k+1)} x^n\right]$; $y_2 = b_0 \left[1 + \sum_{n=1}^{\infty} \dfrac{(-1)^n}{n!\prod_{k=1}^{n}(4k-1)} x^n\right]$

3. (a) $r_1 = 2$, $r_2 = -1$; $y_1 = \sum_{n=0}^{\infty} a_n x^{n+2}$; $y_2 = A y_1 \ln(x) + \sum_{n=0}^{\infty} b_n x^{n-1}$;

(b) $a_n = -\dfrac{n+1}{n(n+3)} a_{n-1}$

(c) $y_1 = a_0 x^2 \left(1 - \dfrac{1}{2}x + \dfrac{3}{20}x^2 - \dfrac{1}{30}x^3 + ...\right)$; $y_2 = b_0 \left(\dfrac{1}{x} - \dfrac{1}{2}\right)$

4. (a) $r_1 = 1$, $r_2 = -1$; $y_1 = \sum_{n=0}^{\infty} a_n x^{n+1}$; $y_2 = A y_1 \ln(x) + \sum_{n=0}^{\infty} b_n x^{n-1}$;

(b) $a_n = \dfrac{1}{n(n+2)} a_{n-1}$

(c) $y_1 = a_0 \sum_{n=0}^{\infty} \dfrac{1}{(n!)(n+2)!} x^{n+1}$; $y_2 = 3y_1 \ln(x) - b_0 x^{-1} \left(1 - x + \dfrac{1}{12}x^3 + \dfrac{13}{960}x^4 + ...\right)$

5. (a) $r_1 = r_2 = -2$; $y_1 = \sum_{n=0}^{\infty} a_n x^{n-2}$; $y_2 = y_1 \ln(x) + \sum_{n=0}^{\infty} b_n x^{n-2}$;

(b) $a_n = -\dfrac{1}{n^2} a_{n-1}$

(c) $y_1 = a_0 \sum_{n=0}^{\infty} \dfrac{(-1)^n}{(n!)^2} x^{n-2}$; $y_2 = y_1 \ln(x) + b_0 \left(\dfrac{2}{x} - \dfrac{3}{4} + \dfrac{11}{108}x - \dfrac{25}{576}x^2 + ...\right)$

6. (a) $r_1 = r_2 = 2$; $y_1 = \sum_{n=0}^{\infty} a_n x^{n+2}$; $y_2 = y_1 \ln(x) + \sum_{n=0}^{\infty} b_n x^{n+2}$;

(b) $a_n = \dfrac{n+1}{n^2} a_{n-1}$

(c) $y_1 = a_0 \sum_{n=0}^{\infty} \dfrac{n+1}{n!} x^{n+2}$; $y_2 = y_1 \ln(x) - b_0 x^3 \left(3 + \dfrac{13}{4}x + \dfrac{31}{18}x^2 + \dfrac{173}{288}x^3 + ...\right)$