THE RIEMANN HYPOTHESIS

PART I
THE PRIME NUMBER THEOREM

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JULY 29, 2010
“book report”
“PRIME OBSESSION”
PRESENTATION OF CREDENTIALS
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—F. W. Bessel 1810
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—F. W. Bessel 1810
C. F. Gauss 1799
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   —F. W. Bessel 1810
   C. F. Gauss 1799
Prologue

• Is there a general rule or formula for how many primes there are less than a given quantity, *that will spare us the trouble of counting them*? (The Prime Number Theorem PNT, proved in 1896, does this approximately; the Riemann Hypothesis RH, still unproven, does this exactly.)

• The Riemann Hypothesis is now the great white whale of mathematical research. The entire twentieth century was bracketed by mathematicians’ preoccupation with it.
• Unlike the Four-Color Theorem, or Fermat’s Last Theorem, the Riemann Hypothesis is not easy to state in terms a non-mathematician can easily grasp. (The four-color problem was stated in 1852 and solved in 1976; Fermat’s Last ‘Theorem’ was stated in 1637 and solved in 1994; the Riemann Hypothesis was stated in 1859 and remains unsolved to this day.)

• The odd-numbered chapters contain mathematical exposition. The even-numbered chapters offer historical and biographical background matter. Chapters 1-10 constitute Part I: The Prime Number Theorem; chapters 11-22 constitute Part II: The Riemann Hypothesis.
Chapter 1—Card Trick

I.

Like many other performances, this one begins with a deck of cards. Take an ordinary deck of 52 cards, lying on a table, all four sides of the deck squared away. Now, with a finger slide the topmost card forward without moving any of the others. How far can you slide it before it tips and falls? Or, to put it another way, how far can you make it overhang the rest of the deck?

First Card \( \frac{1}{2} \) half of one card overhang is \( \frac{1}{2} \) of one card length
Second Card

\[ \frac{3}{4} + \frac{1}{4} \]

half of two cards

overhang is \( \frac{1}{2} + \frac{1}{4} \) of one card length
Third Card

\[
\frac{1}{6} + \left( \frac{1}{6} + \frac{1}{4} \right) + \left( \frac{1}{6} + \frac{1}{4} + \frac{1}{2} \right) = 1\frac{1}{2}
\]
(half of three cards)

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{6} = \frac{11}{12} \text{ card lengths}
\]

total overhang with three cards
Fourth card adds $1/8$ to the overhang
(overhang now more than one full card length)
for the 51 cards you push. (No point pushing the very bottom one.)
This comes out to a shade less than 2.25940659073334. So you have a
total overhang of more than two and a quarter card lengths! (See
Figure 1-6.)

I was a college student when I learned this. It was summer vaca-
tion and I was prepping for the next semester’s work, trying to get
ahead of the game. To help pay my way through college I used to
spend summer vacations as a laborer on construction sites, work that
was not heavily unionized at the time in England. The day after I
found out about this thing with the cards I was left on my own to do
some clean-up work in an indoor area where hundreds of large,
square, fibrous ceiling tiles were being removed. The tiles
were

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{102}
\]

total overhang using all 52 cards
\approx 2.259406 \text{ card lengths}
It is a mathematician's first imperative to **EXTRAPOLATE!**

With 100 cards the total overhang is

\[
\frac{1}{2}(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{99}) \approx 2.588
\]

With a trillion cards the total overhang is

\[
\frac{1}{2}(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{[10^9-1]}) \approx 14.10411
\]

(this is about 4 feet)

You can get any length of overhang with an unlimited supply of cards (Why?)
・ Divergence of the harmonic series

\[ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty \]

Proof: (Nicole d'Oresme 1323–1382).

\[
\begin{align*}
\frac{1}{3} + \frac{1}{4} & > \frac{1}{2} \\
\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} & > \frac{1}{2} \\
\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16} & > \frac{1}{2} \\
\cdots
\end{align*}
\]

(“You can’t beat going to the original sources.”)
• The traditional division of mathematics into subdisciplines:
  Arithmetic (whole numbers)
  Geometry (figures)
  Algebra (abstract symbols)
  Analysis (limits).

• The first and last combine to form *analytic number theory*. There are others (For example, set theory, probability, statistics, combinatorics, game theory, dynamical systems, topology, APPLIED MATHEMATICS; and more).
MATHEMATICS SUBJECT CLASSIFICATION
(AMERICAN MATHEMATICAL SOCIETY)

00-XX General
01-XX History and biography
03-XX Mathematical logic and foundations
05-XX Combinatorics
06-XX Lattices, ordered algebraic structures
08-XX General algebraic systems
11-XX NUMBER THEORY
12-XX Field theory and polynomials
13-XX Commutative algebra
14-XX Algebraic geometry
15-XX Linear algebra; matrix theory
16-XX Associative rings and algebras
17-XX Nonassociative rings and algebras
18-XX Category theory; homological algebra
19-XX K-theory
20-XX Group theory and generalizations
22-XX Topological groups, Lie groups
26-XX Real functions
28-XX Measure and integration
30-XX COMPLEX FUNCTION THEORY
31-XX Potential theory
32-XX Several complex variables
33-XX Special functions
34-XX Ordinary differential equations
35-XX Partial differential equations
37-XX Dynamical systems, ergodic theory
39-XX Difference and functional equations
40-XX Sequences, series, summability
41-XX Approximations and expansions
42-XX Harmonic analysis on Euclidean spaces
43-XX Abstract harmonic analysis
44-XX Integral transforms
45-XX Integral equations
46-XX Functional analysis
47-XX Operator theory
49-XX Calculus of variations, optimal control
51-XX Geometry
52-XX Convex and discrete geometry
53-XX Differential geometry
54-XX General topology
55-XX Algebraic topology
57-XX Manifolds and cell complexes
58-XX Global analysis, analysis on manifolds
60-XX Probability theory
62-XX Statistics
65-XX Numerical analysis
68-XX Computer science
70-XX Mechanics of particles and systems
74-XX Mechanics of deformable solids
76-XX Fluid mechanics
78-XX Optics, electromagnetic theory
80-XX Classical thermodynamics, heat
81-XX Quantum theory
82-XX Statistical mechanics, matter
83-XX Relativity and gravitational theory
85-XX Astronomy and astrophysics
86-XX Geophysics
90-XX Operations research
91-XX Game theory, economics
92-XX Biology and other natural sciences
93-XX Systems theory; control
94-XX Information and communication
97-XX Mathematics education
Chapter 2—The Soil, the Crop

- Riemann (1826–1866) does not seem to have been a good scholar. He had the type of mind that could hold only those things it found interesting, mathematics mostly.

- At some point during the year 1847 Riemann must have confessed to his father that he was far more interested in math than in theology and his father, who seems to have been a kind parent, gave his consent to mathematics as a career.
• Riemann was extremely shy, very pious, thought deeply about philosophy, and was a hypocondriac never in good health.

• Outwardly he was pitiable; inwardly, he burned brighter than the sun.
• The year 1857 was Riemann’s “breakout year.” His 1851 doctoral dissertation is nowadays regarded as a classic of 19th century mathematics, and his 1857 paper was at once recognized as a major contribution.

• In 1859 he was promoted to full professor at Göttingen, on which occasion he submitted a paper titled “On the Number of Prime Numbers Less Than a Given Quantity.” Mathematics has not been quite the same since.

• Riemann’s scientific achievements, which include fundamental contributions to arithmetic, geometry, and analysis, took place during the period 1851-1866.
Chapter 3—The Prime Number Theorem

• Is there a biggest prime? NO (300BCE).

• Whole numbers are to primes what molecules are to atoms (Fundamental Theorem of Arithmetic). Atoms run out before you get to 100; the primes go on forever.

• Do the primes eventually thin out. Can we find a rule, a law, to describe the thinning-out? There are

  25 primes between 1 and 100
  17 between 401 and 500
  14 between 901 and 1000
  4 between 999,901 and 1,000,000.

• The Prime Counting Function; overloading a symbol. The number of primes up to a given quantity $x$ is denoted by $\pi(x)$ ($x$ need not be a whole number).
• The Prime Number Theorem states roughly that: $\pi(N)$ behaves very much like $N/\log N$. Empirically, if you compare $N$ with $N/\pi(N)$, each time $N$ is multiplied by 1000, $N/\pi(N)$ goes up by $\log 1000 = \text{about } 6.9$ (This is the ‘natural logarithm’, to base $e = 2.718 \cdots$, not to base 10.)

• PNT was conjectured by Gauss at the end of the 18th century, and proved by two mathematicians (independently and simultaneously) at the end of the 19th century, using tools developed by Riemann in the middle of the 19th century.

• If the Riemann Hypothesis is true, it would lead to an exact formulation of PNT, instead of one that is always off by several percent. Otherwise, the world would be a very different place.
• Two consequences of PNT:

(1) the probability that the \( N \)-th natural number is prime is approximately \( 1 / \log N \);

(2) The \( N \)-th prime number is approximately of size \( N \log N \).

To illustrate the second statement, the following table implies that 41 = 13 log 13 (approximately), 61 = 19 log 19 (approximately) and so forth.

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Chapter 4—On the Shoulders of Giants

• The greatest mathematician who ever lived was the first person to whom the truth contained in the PNT occurred—Carl Friedrich Gauss (1777-1855).

• Theorems and proofs that would have made another man’s reputation, Gauss left languishing in his personal diaries. (So much to do; so little time!)
• Two anecdotes about Gauss

(age 10) $1 + 2 + \cdots + 100 = ?$

(age 15) the word *chiliad*.

“I have very often spent an idle quarter of an hour to count another chiliad here and there; but I gave it up at last without quite getting through a million.”

Beginning in 1792, at the age of 15, Gauss had amused himself by tallying all the primes in blocks of 1,000 numbers (chiliads) at a time, continuing up into the high hundreds of thousands.
• The other first rank mathematical genius born in the 18th century—Leonhard Euler (1707-1783)—solved the “Basel problem” (chapter 5) and discovered the “Golden Key” (chapter 7).

• There is also the ‘Russian connection’: Peter the great established an Academy in St. Petersburg in 1682 and imported Euler from Switzerland to run it—Russia had just come out of a dark period of its development
Chapter 5—Riemann’s Zeta Function

The Basel problem opens the door to the zeta function, which is the mathematical object the Riemann Hypothesis is concerned with.

What is the Basel problem?

Consider first some infinite series.

\[ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots \text{ (diverges)} \]
\[ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \cdots \text{ (converges, } = 2) \]
\[ 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} \cdots \text{ (converges, } = \frac{3}{2}) \]
\[ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots \text{ (} = \log 2 \text{)} \]
\[ 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} \cdots \text{ (} = \frac{2}{3} \text{)} \]
\[ 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} \cdots \text{ (} = \frac{3}{4} \text{)} \]
The Basel problem (posed in 1689) is:
What is the exact value of
\[ 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \cdots ? \]
The answer (Euler 1735): \( \pi^2/6 \).

He also showed that
\[ 1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} \cdots = \pi^4/90 \]
\[ 1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} \cdots = \pi^6/945, \text{ and so forth.} \]

In summary, Euler found the exact value of
\[ 1 + \frac{1}{2^N} + \frac{1}{3^N} + \frac{1}{4^N} \cdots \text{ for every even } N = 2, 4, 6, \ldots. \]

However, to this day, no one knows the exact value of this series for any odd value of \( N \), \( N = 3, 5, 7, \ldots. \)

Are they irrational?
• Replace the exponent \( N = 2 \) in the Basel problem by any (for the moment real) number \( s \) to get the zeta function

\[
\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.
\]

• The series defining the zeta function converges as long as \( s > 1 \) (\( s \) need not necessarily be a whole number) but it diverges for \( s = 1 \). It appears that the zeta function also diverges for any \( s < 1 \) (since the terms are bigger than the corresponding terms for \( s = 1 \)) and it behaves like \( 1/(s - 1) \) for \( s > 1 \).

Thus, the domain of the zeta function is the set of all (real) numbers greater than 1. Right?

WRONG! (Chapter 9)
Chapter 6—The Great Fusion

• The Riemann Hypothesis was born out of an encounter between “counting logic,” and “measuring logic.” It arose when some ideas from arithmetic (counting) were combined with some from analysis (measuring) to form a new branch of mathematics, analytic number theory. The great fusion between arithmetic and analysis came about as the result of an inquiry into prime numbers.

• Statement of the Riemann Hypothesis:

    All non-trivial zeros of the zeta function have real part one-half.
• Analysis dates from the invention of calculus by Newton and Leibnitz in the 1670s.

• Arithmetic, by contrast with analysis, is widely taken to be the easiest, most accessible branch of math. Be careful though—it is rather easy to state problems that are ferociously difficult to prove (e.g., Goldbach conjecture, Fermat’s Last ‘Theorem’).
• The Goldbach conjecture asks if every even integer bigger than 2 can be written as the sum of two (odd) primes. This is yet another still unsolved problem with absolutely no real-life application.

• A problem that can be stated in a few plain words, yet which defies proof by the best mathematical talents for decades (centuries!), has an irresistible attraction for most mathematicians. Even failed attempts can generate powerful new results and techniques. And there is, of course, the Mallory factor.
• Euler proved the Golden Key in 1737. One hundred years later, it came to the attention of Dirichlet, who combined it with Gauss’s work on congruences (Clock arithmetic!) to answer an important question about prime numbers, generally considered to be the beginning of analytic number theory (How many primes are there in an arithmetic progression?).
Gauss and Dirichlet were Riemann’s two mathematical idols. If it was Riemann who turned the key, it was Dirichlet who first showed it to him and demonstrated that it was a key to something or other; and it is to Dirichlet that the immortal glory of inventing analytic number theory properly belongs.

WHAT THEN, EXACTLY, IS THIS GOLDEN KEY?
Chapter 7—The Golden Key, and an Improved Prime Number Theorem

- Both primes and the zeta function were of interest to Riemann. By yoking the two concepts together, by turning the Golden Key, Riemann opened up the whole field of analytic number theory.

- The Golden Key is just a way that Euler found to express the sieve of Eratosthenes (230 BCE) in the language of analysis. ("You can't beat going to the original sources.") The Golden Key expresses the Zeta-function, a sum involving all positive integers, in terms of a product involving only prime numbers.
\[ \zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \]

\[ \frac{1}{2^s} \zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} \cdots \]

\[ (1 - \frac{1}{2^s}) \zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} \cdots \]

\[ (1 - \frac{1}{3^s})(1 - \frac{1}{2^s}) \zeta(s) = \]

\[ = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \frac{1}{13^s} \cdots + \frac{1}{25^s} + \cdots \]

\[ \cdots \]

\[ \prod_p (1 - \frac{1}{p^s}) \zeta(s) = 1 \]

**VOILÀ LA GOLDEN KEY!**

\[ \sum_n n^{-s} = \prod_p (1 - p^{-s})^{-1} \]
• The significance of the Golden Key will be not seen until it is “turned”. To prepare for this turning, a little bit of calculus is needed, namely, the basics of differentiation and integration!
• An important function, called the “log integral function,” and denoted by $Li(x)$, is defined as the area from 0 to $x$ under the graph of $1/\log t$.

• The PNT states that $\pi(N)$ behaves very much like $N/\log N$. It is also true that $Li(N)$ behaves very much like $N/\log N$. The improved PNT states that $\pi(N)$ behaves much more like $Li(N)$ than it does like $N/\log N$. The exact formula for $\pi(x)$ (stated by Riemann in 1859, still unproven) leads off with $Li(x)$.

• Up to at least $N = 100$ trillion, $Li(N)$ is larger than $\pi(N)$. Is $Li(x)$ always bigger than $\pi(x)$? Surprisingly, NO!
Chapter 8—Not Altogether Unworthy

- Anyone who wanted to do serious mathematics in the 1840s needed to be in Paris or Berlin.

- Riemann studied for two years in Berlin under Dirichlet, but returned to Göttingen in 1849 to pursue his doctorate under Gauss.

- After Riemann’s final examination, Gauss drooled: “A substantial and valuable work, which does not merely meet the standards required for a doctoral dissertation, but far exceeds them. Riemann’s doctoral thesis is a masterpiece.”
• In Riemann’s doctoral dissertation, the “Riemann integral” occurs, now taught as a fundamental concept in calculus courses.

• His habilitation lecture (second doctoral degree) was on the foundations of geometry. The ideas contained in this paper were so advanced that it was decades before they became fully accepted, and 60 years before they found their natural physical application, as the mathematical framework for Einstein’s General Theory of Relativity.
• From the death of Gauss to the death of Dirichlet was four years, two months, and twelve days. In that span, Riemann lost not only the two colleagues he had esteemed above all other mathematicians, but also his father, his brother, and two of his sisters.

• During this time, Riemann’s star in the world of mathematics had been rising. It was therefore not very surprising that the authorities selected Riemann as the second successor of Gauss’s professorship in 1859. Two weeks later, he was appointed a corresponding member of the Berlin Academy, leading to his famous paper containing his Hypothesis.
• Pafnuty Chebyshev in St. Petersburg made two significant contributions between Dirichlet’s picking up the Golden Key in 1837 and Riemann’s turning it in 1859.
(1) In 1849, a conditional PNT (the condition was removed a half century later!), namely,

\[ \pi(N) \sim \frac{CN}{\log N} \text{ for some fixed number } C, \text{ then } C = 1. \]

(2) In 1850, “Bertrand’s postulate,” namely,

Between any number and its double, there is guaranteed to be at least one prime number.

A second result in 1850 was a crude estimate for the difference between \( \pi(N) \) and \( N/\log N \), namely, they cannot differ by more than about 10%.
Chebyshev’s methods were elementary, Riemann’s were not, nor were the original proofs of PNT (1896). An elementary proof of PNT was not produced until 1949 (Atle Selberg and Paul Erdös).

One point of interest of this chapter is the fact that Chebyshev’s name, like that of Shakespeare, has numerous spellings, so is a data retrieval nightmare.
Chapter 9—Domain Stretching

- The Riemann Hypothesis states: All non-trivial zeros of the zeta function have real part one-half.

- What is a zero of a function? What are the zeros of the zeta function? When are they non-trivial. After we answer these questions we’ll move on to “real part one-half.”

- A “zero” of a function is a number \( a \) such that the function has the value zero at \( a \). In other words, if you graph the function, its zeros are the numbers on the \( x \)-axis at which the function touches crosses the \( x \)-axis. A good example is the function \( \sin x \), which has zeros at \( x = 0, \pi, -\pi, 2\pi, -2\pi, \ldots \)
• An infinite series might define only part of a function; in mathematical terms, an infinite series may define a function over only part of that function’s domain. The rest of the function might be lurking away somewhere, waiting to be discovered by some trick.

**EXAMPLE 1:** $S(x) = 1 + x + x^2 + x^3 + \cdots$, which converges for $-1 < x < 1$ and equals $1/(1-x)$ for those values of $x$. Since $1/(1-x)$ makes sense for all numbers except $x = 1$, this shows that the domain of $S(x)$ is larger than $-1 < x < 1$.

**EXAMPLE 2:** The Gamma function and the factorial symbol. If you define $H(x) = \int_0^\infty e^{-t} t^{x-1} \, dt$, then one has $H(2) = 1$, $H(3) = 2$, $H(4) = 6$, $H(5) = 24, \ldots$, in fact for every positive integer $m$, $H(m) = (m - 1)! = 1 \cdot 2 \cdot 3 \cdots \cdot (m - 2)(m - 1)$. 
It is a mathematician’s second imperative to **EXTEND!**

- In addition to arguments greater than 1, the zeta function has values for all arguments less than 1. This extension of the zeta function is done in two steps: first to all arguments between 0 and 1 (by changing signs in the series), and then to all negative arguments (by using a deep formula in Riemann’s famous 1859 paper).

- The extended zeta function has the value zero at every negative even number. These are the trivial “zeros” of the zeta function.
STEP 1: Define the “eta” function:

\[ \eta(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} \cdots \]

This converges whenever \( s > 0 \).

It is easy to see that for \( s > 1 \),

\[ \zeta(s) = \frac{\eta(s)}{1 - \frac{1}{2^{s-1}}} \]

Therefore, \( \zeta(s) \) extends to the positive real axis.
STEP 2: From Riemann’s 1859 paper, for every $s > 0$ (except for $s = 1$)

$$
\zeta(1 - s) = 2^{1-s} \pi^{-1} \sin\left(\frac{1-s}{2} \pi\right)(s-1)! \zeta(s)
$$

For example $\zeta(-15)$ can be calculated from $\zeta(16)$

This extends $\zeta(s)$ to all values of $s < 0$ and moreover shows that if $m = 1, 2, \ldots$

$$
\zeta(-2m) = 0
$$

Thus $\zeta(s)$ is defined for all real number except for $s = 0, 1$ and the negative even integers

$$
-2, -4, -6, -8, \ldots
$$

are what are called the “trivial zeros” of the zeta function.
Chapter 10—A Proof and a Turning Point

- Riemann had a strongly visual imagination. His mind leaped to results so powerful, elegant, and fruitful that he could not always force himself to pause to prove them.

- The 1859 paper is therefore revered not for its logical purity, and certainly not for its clarity, but for the sheer originality of the methods Riemann used, and for the great scope and power of his results, which have provided, and will yet provide, Riemann’s fellow mathematicians with decades of research.
• If the Riemann Hypothesis were true, it would reveal a deep secret about prime numbers which has no foreseeable practical consequences that could change the world. In particular, PNT would follow as a consequence. However, RH is much stronger than PNT, and the latter was proved using weaker tools.

• There were several significant landmarks between Riemann’s paper in 1859 and the proof of PNT (in 1896). The main significance of Riemann’s paper for the proof of the PNT is that it provided the deep insights into analytic number theory that showed the way to a proof.
• One of the landmarks alluded to above was the following: In 1895 the German mathematician Hans von Mangoldt proved the main result of Riemann’s paper, which states the connection between \( \pi(x) \) and the zeta function. It was then plain that if a certain assertion much weaker than the Riemann Hypothesis could be proved, the application of the result to von Mangoldt’s formula would prove the PNT.
The PNT follows from a much weaker result (than RH), which has no name attached to it: All non-trivial zeros of the zeta function have real part less than one. This result was proved in 1896 simultaneously and independently by a Frenchman Jacques Hadamard and a Belgian Charles de la Vallée Poussin. This established the PNT using the Riemann-von Mangoldt formula.
• If the PNT was the great white whale of number theory in the 19th century, RH was to take its place in the 20th, and moreover was to cast its fascination not only on number theorists, but on mathematicians of all kinds, and even on physicists and philosophers.

• There is also the neat coincidence of the PNT being first thought of at the end of one century (Gauss, 1792), then being proved at the end of the next (Hadamard and de la Vallée Poussin, 1896).

• The attention of mathematicians turned to RH, which occupied them for the following century—which came to its end without any proof being arrived at.
By the later 19th century the world of mathematics had passed out of the era when really great strides could be made by a single mind working alone. Mathematics had become a collegial enterprise in which the work of even the most brilliant scholars was built upon, and nourished by, that of living colleagues.

One recognition of this fact was the establishment of periodic International Congresses of Mathematicians, with PNT among the highlights of the first meeting in 1897 in Zürich. There was a second Congress in Paris in 1900.
The Paris Congress will forever be linked with the name of David Hilbert, a German mathematician working at Göttingen, the university of Gauss, Dirichlet, and Riemann, for his address on the mathematical challenges of the new century, RH being the most prominent among them.

- Each problem came from some key field of mathematics at the time. If they were to be solved, their solution would advance that field in new and promising directions.
Chapter 11—Nine Zulu Queens Ruled China

• We know what the trivial zeros of the zeta function are. What are the non-trivial zeros? For this we need to know about complex numbers.

• Mathematicians think of numbers as a set of nested Russian dolls. The inhabitants of each Russian doll are honorary inhabitants of the next one out.

• In $\mathbb{N}$ you can’t subtract; in $\mathbb{Z}$ you can’t divide; in $\mathbb{Q}$ you can’t take limits; in $\mathbb{R}$ you can’t take the square root of a negative number. With the complex numbers $\mathbb{C}$, nothing is impossible. You can even raise a number to a complex power.
• Therefore, in the zeta function, the variable \( s \) may now be a complex number, and the Riemann hypothesis now makes sense: it asserts that the non-trivial zeros of the zeta function all lie on the vertical line whose horizontal coordinate is equal to 1/2.

• We shall need a complex plane extension process to determine the precise domain of this complex valued zeta function.
Chapter 12—Hilbert’s Eighth Problem

• Since 1896 it was known, with mathematical certainty, that, yes indeed, \( \pi(N) \) could be approximated arbitrary closely by \( N/ \log N \). Everyone’s attention now focused on the nature of the approximation—What is the error term?

• Riemann did not prove the PNT, but he strongly suggested it was true, and even suggested an expression for the error term. That expression involved all the non-trivial zeros of the zeta function.

• One of the questions before us is: what exactly is the relation between the zeros of the zeta function and the prime number theorem. We shall answer this in Part II (September 14, 2010).

END OF PART I
LAYMAN’S SUMMARY OF WHAT WE LEARNED

There is a mathematical expression that predicts roughly how many prime numbers there are smaller that any number you care to name. You know also that this prediction, by Gauss, is not entirely accurate, and that the amount by which it is wrong is the subject of another mathematical expression, devised by the German mathematician Bernhard Riemann. With Gauss’s estimate, proved independently by two other mathematicians in 1896, and Riemann’s correction, conjectured but not yet proved by anyone, we know much more about how the prime numbers are distributed. At the heart of Riemann’s correction factor, and essential to understanding how it is related to prime numbers, is Riemann’s zeta function, and in particular, a series of numbers which are known as the Riemann zeros.
The popular idea of mathematics is that it is largely concerned with calculations. In fact, “mathematics is no more the art of reckoning and computation that architecture is the art of making bricks or hewing wood, no more than painting is the art of mixing colors on a palette, no more that the science of geology is the art of breaking rocks, or the science of anatomy the art of butchering.”