CONTENT  This exam will cover the material discussed in Chapter 9.

TOPICS  You should be comfortable with the following topics:

Finding the limit of a sequence, recursively defined sequence, finding the \( n \)th term of a sequence, simplifying factorials, partial sum of series, the \( n \)th-term test for divergence, telescoping series, geometric series, sum of a geometric series, the integral test, \( p \)-series test, direct comparison test, limit comparison test, alternating series test, absolute and conditional convergence, the remainder theorem for an alternating series, ratio test, root test, Taylor and Maclaurin polynomials, remainder of a Taylor polynomial, power series, interval and radius of convergence, derivative and integral of a power series, Taylor and Maclaurin series.

FORMULAS  You should have the following formulas memorized.

**Sum of a Geometric Series**
\[
\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad |r| < 1
\]

**Taylor Coefficient**
\[
f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \iff a_n = \frac{f^{(n)}(c)}{n!}
\]

**Taylor Polynomial Remainder**
\[
R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} (x-c)^{n+1}
\]
where \( z \) is between \( x \) and \( c \)

**The Taylor Polynomial**
\[
P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n
\]

**Common Taylor Series**
\[
\frac{1}{1 + x} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \quad (-1, 1)
\]
\[
\ln(x + 1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}, \quad (-1, 1]
\]
\[
\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!}, \quad (-\infty, \infty)
\]
\[
\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad (-\infty, \infty)
\]
\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad (-\infty, \infty)
\]
1. (9.1) Find the limit of the sequence, if it exists.

(a) \( a_n = \frac{2 - 3n}{\sqrt{n^2 + 1}} \)
(b) \( a_n = \sqrt{n} \)
(c) \( a_n = 0, 1 - \frac{1}{2}, 1 - \frac{1}{3}, 1 - \frac{1}{4}, \ldots \)

2. (9.1) Find the \( n \)th term of the sequence.

(a) \( 0, \frac{1}{\sqrt{2}}, \frac{2}{\sqrt{6}}, \frac{3}{\sqrt{12}}, \frac{2}{\sqrt{6}}, \frac{5}{\sqrt{12}}, \ldots \)
(b) \( \frac{1}{2}, -\frac{3}{5}, \frac{1}{2}, -\frac{7}{17}, \frac{9}{25}, -\frac{11}{37}, \ldots \)

3. (9.2) Find the sum of the convergent series.

(a) \( \sum_{n=1}^{\infty} \frac{4}{n(n+4)} \)
(b) \( 64 - 16 + 4 - 1 + \ldots \)

4. (9.2) The first pass of a pendulum spans 45° and each subsequent pass spans 99.8% of the previous. If the pendulum continues swinging following this pattern, find the total accumulated distance traveled by a point 8 inches from the pivot point. Hint: Arc length is \( S = \theta r \) where \( \theta \) is in radians.

5. (9.5) Determine whether the given series is absolutely convergent, conditionally convergent, or divergent.

(a) \( \sum_{n=0}^{\infty} \frac{(-2)^n}{3^n + 1} \)
(b) \( \sum_{n=1}^{\infty} \frac{\sin \left( \frac{\pi}{2} (2n - 1) \right)}{n} \)
(c) \( \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{e^2(n+1)} \)

6. (9.5) Find the number of terms necessary to approximate the sum of the series with an error less than 0.0001.

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot 4^n} = \ln \left( \frac{5}{4} \right) \]

7. (9.2 - 9.6) Show the convergence or divergence of each series. Use each of the ten methods learned in Chapter 9 at least once.

(a) \( \sum_{n=1}^{\infty} \frac{1}{n^6} \)
(b) \( \sum_{n=0}^{\infty} \frac{7}{0.1^n} \)
(c) \( \sum_{n=0}^{\infty} \frac{n}{2^n} \)
(d) \( \sum_{n=0}^{\infty} \frac{n!}{3n - 1} \)
(e) \( \sum_{n=1}^{\infty} \frac{(n + 1)!}{n^2} \)
(f) \( \sum_{n=1}^{\infty} \frac{4n^3 \sqrt{n}}{6n^2 + 5} \)
(g) \( \sum_{n=0}^{\infty} \frac{(-1)^nn}{n^2 - 1} \)
(h) \( \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \)
(i) \( \sum_{n=1}^{\infty} \frac{1}{n - \cos^2(n)} \)
(j) \( \sum_{n=1}^{\infty} \frac{1}{(2n + 1)(2n - 1)} \)
(k) \( \sum_{n=1}^{\infty} \left( \frac{3n}{5n + 1} \right)^n \)
(l) \( \sum_{n=1}^{\infty} ne^{-n^2} \)

8. (9.7) Find the first three nonzero terms of the Maclaurin polynomial for \( f(x) = \sin \pi x \).

9. (9.7) Find the 4th degree Taylor polynomial for \( f(x) = \ln(2x - 1) \), centered at \( c = 1 \).

10. (9.7) Use Taylor’s Theorem to obtain an upper bound for the error in the third degree Maclaurin polynomial approximation of \( e^x \) evaluated at \( x = 0.5 \). Then calculate the exact value of the error.

11. (9.8) Find the radius of convergence of the power series.

(a) \( \sum_{n=0}^{\infty} \frac{\pi^n(x - 1)^{2n}}{(2n + 1)!} \)
(b) \( \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^n(n + 1)} \)
12. (9.8) Find the interval of convergence of the power series.

(a) \( \sum_{n=0}^{\infty} \frac{x^n}{n+1} \)  
(b) \( \sum_{n=0}^{\infty} \left( \frac{3}{4} \right)^n (x + 5)^n \)

13. (9.9) Find a power series for the function, centered at \( c = 2 \), and find the interval of convergence.

\( f(x) = \frac{1}{2x + 5} \)

14. (9.9) Use the power series for \( \frac{1}{1 + x} \), to determine a power series, centered at \( c = 0 \) for the function \( f(x) = \ln(1 + x^2) \). Identify the interval of convergence.

15. (9.10) Find the Maclaurin series for \( \sin(x^2) \) using the series for \( \sin x \). Express your answer in \( \Sigma \) notation.

16. (9.10) Use the first three terms of your answer for the previous question to approximate the value (to six decimal places) of the integral.

\( \int_{0}^{1} \sin(x^2) \, dx \)

17. (9.10) Find the sum of the series: \( \sum_{n=2}^{\infty} \frac{3^n}{n!} \).
1. (a) $-3$
   (b) $1$
   (c) $1$

2. (a) $a_n = \frac{n-1}{\sqrt{n}!}$
   (b) $a_n = (-1)^{n+1} \frac{2n-1}{n^2 + 1}$

3. (a) $\frac{25}{12}$
   (b) $\frac{256}{5}$

4. $1000\pi \approx 3141$ inches

5. (a) Absolutely convergent
   (b) Conditionally convergent
   (c) Conditionally convergent

6. Five terms

7. Methods indicated are suggestions only. Your method may vary.
   (a) Converges (p-series)
   (b) Diverges (geometric series)
   (c) Converges (root test)
   (d) Diverges (nth term test)
   (e) Diverges (ratio test)
   (f) Converges (limit comparison)
   (g) Diverges (alternate series)
   (h) Converges (integral test)
   (i) Diverges (direct comparison)
   (j) Converges (telescoping)
   (k) Converges (root test)
   (l) Converges (integral test)

8. $P_5 = \pi x - \frac{\pi^3 x^3}{3!} + \frac{\pi^5 x^5}{5!}$

9. $P_4 = 2(x-1) - 2(x-1)^2 + \frac{8}{3}(x-1)^3 - 4(x-1)^4$

10. $R_3 \leq 0.004294$; Actual error $\approx 0.002888$

11. (a) $R = \infty$
    (b) $R = 3$

12. (a) $[-1, 1]$
    (b) $\left( -\frac{19}{3}, -\frac{11}{3} \right)$

13. $\sum_{n=0}^{\infty} (-1)^n \frac{2^n (x-2)^n}{9n+1}$, on $\left( -\frac{5}{2}, \frac{13}{2} \right)$

14. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{n+1}$, on $[-1, 1]$

15. $\sum_{n=0}^{\infty} (-1)^n x^{4n+2} \frac{1}{(2n+1)!}$

16. $0.310281$

17. $e^3 - 4$