Southern California Math Mini-conference for Community College Students

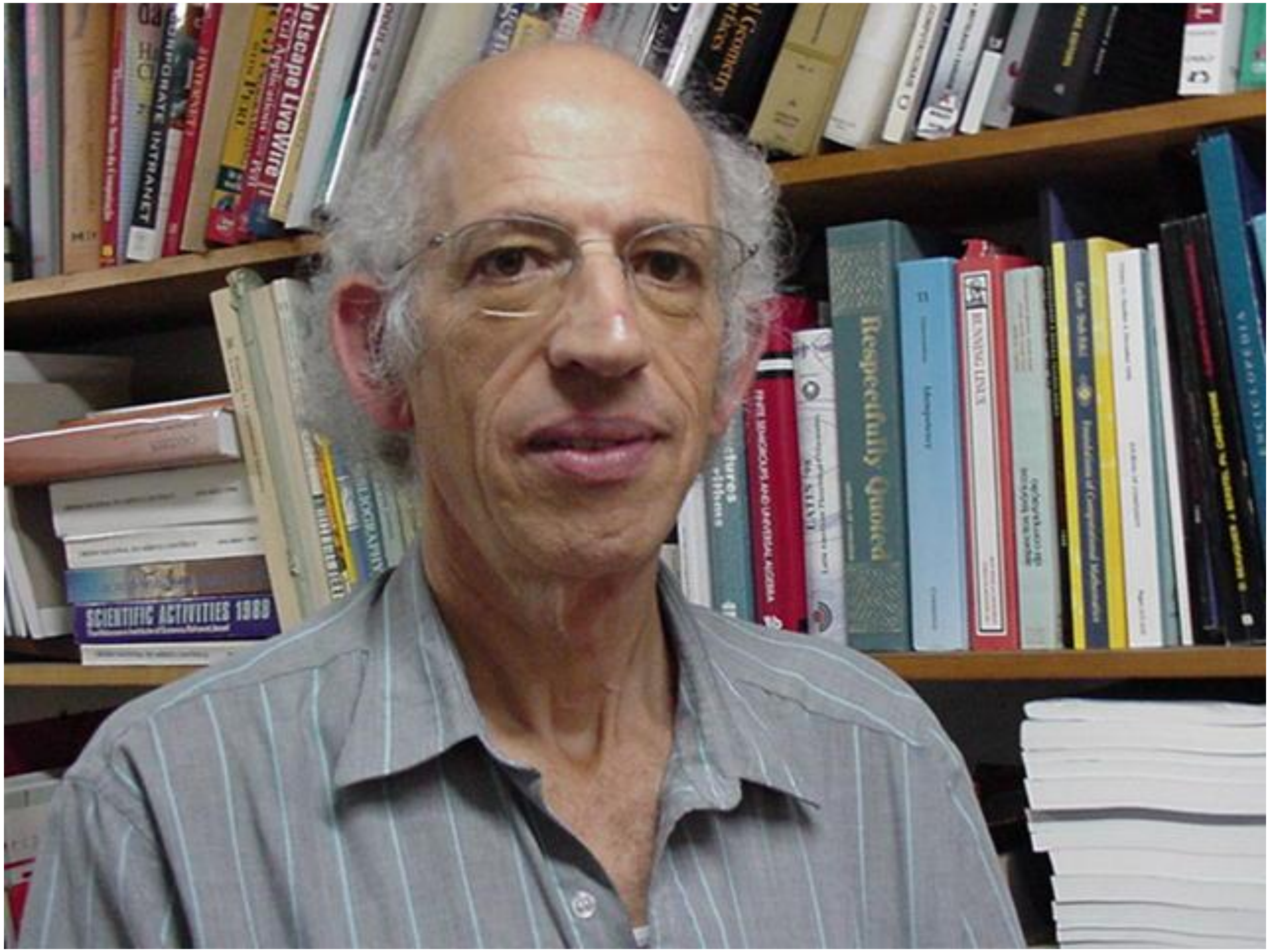
INTRODUCTION TO TROPICAL MATHEMATICS

Objectives

- Origin of “Tropical” Mathematics
- Applications
- Definitions and Arithmetic operations
- Monomials
- The Freshman Dream
- Possible research project

Origin of “Tropical” Mathematics

Origin of “Tropical” Mathematics



Brazilian dude Imre Simon

Applications of Tropical Mathematics

- Geometric Combinatorics
- Algebraic Geometry
- Mathematical Physics
- Number Theory
- Symplectic Geometry
- Computational Biology

Tropical Mathematics

Tropical mathematics is the study of the tropical semiring:

$$(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$$

Ring Definition

Ring: An algebraic structure consisting of:

- A set
- Together with two binary operations (usually addition and multiplication)

With the following axioms:

- Addition is commutative
- Addition and multiplication are associative
- Multiplication distributes over addition
- There exists an additive identity
- Each element in the set has an *additive inverse*

Semiring Definition

Semiring:

Similar to a ring, but **without** the requirement that each element must have an *additive inverse*.

Arithmetic operations

Tropical sum: $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$

$\oplus$ operator // "tropical sum" operator

$x \oplus y \coloneqq \min\{x, y\}$

$1 \oplus 2 = 1$

$5 \oplus 8 = 5$

Arithmetic operations

Tropical product: $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$

$\odot$ operator // "tropical product" operator

$x \odot y \coloneqq x + y$

$1 \odot 2 = 3$

$5 \odot 8 = 13$

Arithmetic operations

Both addition and multiplication are commutative.

$x \oplus y = y \oplus x$

$4 \oplus 7 = 7 \oplus 4$

$4 = 4$

$x \odot y = y \odot x$

$3 \odot 8 = 8 \odot 3$

$11 = 11$

Arithmetic operations

Order of Operations is the same.

Distributive Law holds for both operations.

$$x \odot (y \oplus z) = x \odot y \oplus x \odot z$$

$$\begin{aligned} 4 \odot (3 \oplus 7) &= 4 \odot 3 \oplus 4 \odot 7 \\ &= 7 \oplus 7 \\ &= 7 \end{aligned}$$

Arithmetic operations

Neutral Elements: $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$

$x + 0 = x$ $x \cdot 1 = x$

$x \oplus \infty = x$ // ∞ is the additive identity.

$4 \oplus \infty = 4$

$x \odot 0 = x$ // remember we're adding here

$4 \odot 0 = 4$

Arithmetic operations

Additive Inverse:

The additive inverse, or opposite, of a number *a* is the number that, when added to *a*, yields zero.

Additive inverse of 7 is −7,

$7 + (-7) = 0$,

The additive inverse of a number is the number's negative.

Arithmetic operations

Additive Inverse:

$7 + -7 = 0$

$7 \oplus -7 = -7$ // min{7, -7}

Additive inverse does **not exist** for the tropical semiring.

Arithmetic operations

Tropical difference: $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$

$11 \ominus 3 = ?$

// impossible.

This really means:

$3 \oplus x = ?$

// min{3, x}; no solution
// x is undefined

Change to: $11 \oplus -3 = -3$

// illegal;
// no additive inverse

Tropical Semiring Definition

Tropical Semiring: $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ An algebraic structure consisting of:

A set $\mathbb{R}$ ✓

Together with two binary operations $\oplus$, $\odot$ ✓

With the following axioms:

- Addition is commutative, ✓
- Addition and multiplication are associative, ✓
- Multiplication distributes over addition, ✓
- There exists an additive identity ✓
- No additive inverse ✓

Monomials

Let $x_1, x_2, x_3, \ldots, x_n$ be variables that represent elements in the tropical semiring $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$.

A *monomial* is any product of these variables, where repetition is allowed.

A monomial represents a function from $\mathbb{R}^n$ to $\mathbb{R}$.

$x_2 \odot x_1 \odot x_3 \odot x_1 \odot x_4 \odot x_2 \odot x_3 \odot x_2 = x_1^2 x_2^3 x_3^2 x_4$

$x_2 + x_1 + x_3 + x_1 + x_4 + x_2 + x_3 + x_2 = 2x_1 + 3x_2 + 2x_3 + x_4.$

//= linear function

∴ Tropical monomials are linear functions with integer coefficients.

The Freshman Dream

Definition:

$(x + y)^3 = x^3 + y^3$

//not true

$(x + y)^3 = x^3 + x^2y + xy^2 + y^3$ //true

But! Freshman Dream holds for Tropical Arithmetic

$(x \oplus y)^3 = x^3 \oplus y^3$ //true

The Freshman Dream

Proof:

$(x \oplus y)^3$

$= (x \oplus y) \odot (x \oplus y) \odot (x \oplus y)$

$= (x \odot x \odot x) \oplus (x \odot x \odot y) \oplus (x \odot y \odot y) \oplus (y \odot y \odot y)$

$= x^3 \oplus x^2 y \oplus xy^2 \oplus y^3$

$= x^3 \oplus y^3$

The Freshman Dream

Example:

$(3 \oplus 2)^3 = 3^3 \oplus 2^3$

$= (3 \oplus 2) \odot (3 \oplus 2) \odot (3 \oplus 2)$

$= (3 \odot 3 \odot 3) \oplus (3 \odot 3 \odot 2) \oplus (3 \odot 2 \odot 2) \oplus (2 \odot 2 \odot 2)$

$= 3^3 \oplus 3^2\, 2 \oplus 3 \odot 2^2 \oplus 2^3$

$= 3^3 \oplus 2^3$

$= 2^3$

$= 6$

Research Problem

Research problem:

The tropical semiring generalizes to higher dimensions: The set of convex polyhedra in $\mathbb{R}^n$ can be made into a semiring by taking $\odot$ as "Minkowski sum" and $\oplus$ as "convex hull of the union." A natural subalgebra is the set of all polyhedra that have a fixed *recession cone C*. If $n = 1$ and $C = \mathbb{R}_{\geq 0}$, this is the tropical semiring.

Develop linear algebra and algebraic geometry over these semirings, and implement efficient software for doing arithmetic with polyhedra when $n \geq 2$.